Stability of Convergence Theorems of the Noor Iteration Method for an Enumerable Class of Continuous Hemi Contractive Mapping in Banach Spaces

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Abstract
The purpose of this is to study the Noor iteration process for the sequence \( \{x_n\} \) converges to a common fix point for enumerable class of continuous hemi contractive mapping in Banach spaces.

Keywords: Stability; Noor iterations; Hemicontractive mapping; Convergence theorem; Continuous pseudocontractive mapping

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Introduction
Let \( E \) be a real Banach space and let \( J \) denote the normalized duality mapping from \( E \) to \( E^* \) defined by
\[
J(x) = \{ f \in E^* : \langle x, f \rangle = \|x\| \|f\| \|x\|^{-1} \|f\|^{-1} \text{ for all } x \in E \}
\]
Where \( E^* \) denotes the dual space of \( E \) and \( \langle \cdot, \cdot \rangle \) denotes the generalization duality pair.

It is well known that if \( E^* \) is strictly convex then \( J \) is single–valued. In the sequel, we shall denote the single–valued duality mapping by \( j \). Let \( K \) be a nonempty closed convex subset of Banach space \( E \) and \( T : K \to K \) be a self-mapping of \( K \).

Definition 3.1: (i) A mapping \( T \) with domain \( D(T) \) and range \( R(T) \) in a Banach space is called pseudocontrative mapping, if for all \( x, y \in D(T) \), there exists \( j(x - y) \in J(x - y) \) such that
\[
\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2
\]
(ii) A mapping \( T \) with domain \( D(T) \) and range \( R(T) \) in \( E \) is called a hemicontractive mapping if \( F(T) \neq \emptyset \) and for all \( x, y \in D(T) \), \( x \neq y \in F(T) \) such that,
\[
\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2
\]
(iii) A mapping \( T : K \to K \) is called \( L \)-Lipschitzian there exists \( L > 0 \) such that
\[
\| Tx - Ty \| \leq L \| x - y \| \quad \text{For all } \ x, \ y \in K
\]

Definition 3.2: If \( \{x_n\}_{n=0}^\infty \) and \( \{\beta_n\}_{n=0}^\infty \) are sequences of real numbers in \([0, 1] \) [2]. For arbitrary \( x \in E \), Let \( \{x_n\}_{n=0}^\infty \) be a Noor iteration defined by,
\[
x_{n+1} = (1 - \beta_n)x_n + \beta_n T q_n
\]
\[
c_n = (1 - \beta_n)x_n + \beta_n T r_n
\]

Lemma 3.3: Let \( E \) be a real uniformly convex Banach space [3]. \( K \) is nonempty closed convex subset of \( E \) and \( T \) a continuous pseudocontractive mapping of \( K \), then \( I - T \) is demiclosed at zero, that is, for all sequences \( \{x_n\} \subseteq K \) with \( x_n \to p \) and \( x_n - T x_n \to 0 \) it follows that \( p = T p \)

Lemma 3.4: Let \( \delta \) be a number satisfying \( 0 < \delta < 1 \) and \( \{\epsilon_n\} \) a positive sequence satisfying \( \lim_{n \to \infty} \epsilon_n = 0 \) [4,5]. Then, for any positive sequence \( \{\nu_n\} \) satisfying: \( \nu_{n+1} \leq \delta \nu_n + \epsilon_n \). It follows that \( \lim_{n \to \infty} \nu_n = 0 \).

Results

Theorem 4.1: Let \( \{T_n\}_{n=0}^\infty \) be defined as above and \( F(T) \neq \emptyset \) and let \( (E, \|\cdot\|) \) be a Banach space, \( T : E \to E \) a self-map of \( E \) with a fixed point \( p \), satisfying the contractive condition
\[
\langle Tx - x, j(x - x') \rangle \leq \|x - x'\|^2 \quad \text{For } x \in E.
\]
Let \( \{x_n\}_{n=0}^\infty \) be a Noor iteration converges to \( p \) and defined by the iteration
\[
\{x_n\}_{n=0}^\infty = \{\sum_{i=0}^{n} \alpha_i\} \quad \text{for any real sequence } \{\alpha_i\}, \{x_0\} \text{ is a real sequence in } (0, 1) \text{ and define as}
\]
\[
\epsilon_n = \|x_{n+1} - (1 - \alpha_n) x_n - \alpha_n T q_n \| \quad \text{Then}
\]
\[
\lim_{n \to \infty} \epsilon_n = 0 \quad \text{exists for all } p \in F;
\]
\[
\lim_{n \to \infty} d(x_n, F) = \inf \{d(x_n, p) : p \in F\};
\]
\[
\{x_n\}_{n=0}^\infty \text{ converges strongly to a common fixed point of } \{T_i\}_{i=0}^\infty \text{ if and only if } \lim_{n \to \infty} d(x_n, F) = 0
\]

Proof: Let \( p \in F \) and \( n \geq 1 \) by 3.1 we choose \( j(x_n - p) \in J(x_n - p) \) such that
\[
\| x_{n+1} - p \|^2 = \|x_{n+1} - p, j(x_{n+1} - p)\|
\]
\[
\| x_{n+1} - p \| \leq \| x_{n+1} \| + \| j(x_{n+1} - p) \| |
\]
\[
= \epsilon_n + \| (1 - \alpha_n) x_n + \alpha_n T q_n - ((1 - \alpha_n) x_n + \alpha_n T q_n) - p \|
\]
\[
= \epsilon_n + \| (1 - \alpha_n) x_n + \alpha_n T q_n - p + \alpha_n T q_n - p \|
\]

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\( \leq e_0 + (1 - \alpha) || x_0 - p + \alpha \beta || Tq_n - p || \)
\( = e_0 + (1 - \alpha) || x_0 - p || + \alpha \beta || p - Tq_n || \)
\( \leq e_0 + (1 - \alpha) || x_0 - p || + \alpha \beta || p - q_n || \)
\( = e_0 + (1 - \alpha) || x_0 - p || + \alpha \beta || q_n - p || \)

For the estimate of in (1) we get
\( ||q_n - p|| \leq ||(1 - \beta_n)x_n + \beta_n Tq_n - p|| \)
\( = ||(1 - \beta_n)x_n + \beta_n Tq_n - ((1 - \beta_n) + \beta_n) p|| \)
\( \leq ||(1 - \beta_n)|| x_n - p || + \beta_n || Tq_n - p || \)
\( = (1 - \beta_n)|| x_n - p || + \beta_n || p - Tq_n || \)
\( \leq (1 - \beta_n)|| x_n - p || + \beta_n a || p - r_n || \)
\( = (1 - \beta_n)|| x_n - p || + \beta_n a || r_n - p || \)

Substituting (2) into (1) gives
\( ||x_{n+1} - p|| \leq e_0 + (1 - \alpha) || x_0 - p || + \alpha || \beta || a || r_n - p || \) (3)

For \( ||r_n - p|| \) in (3) we have, \( ||r_n - p|| \leq ||(1 - \gamma_n)x_n + \gamma_n Tq_n - p|| \)
\( = ||(1 - \gamma_n)|| x_n + \gamma_n Tq_n - ((1 - \gamma_n) + \gamma_n) p|| \)
\( \leq ||(1 - \gamma_n)|| x_n - p || + \gamma_n || Tq_n - p || \)
\( = (1 - \gamma_n)|| x_n - p || + \gamma_n || p - Tq_n || \)
\( \leq (1 - \gamma_n)|| x_n - p || + \gamma_n a || p - x_n || \)
\( = (1 - \gamma_n) || x_n - p || + \gamma_n a || x_n - p || \)

Substituting (4) into (3) and using lemma 3.3
\( = e_0 + (1 - \alpha) || x_0 - p || + \gamma_n a || x_n - p || \)
\( = e_0 + (1 - \alpha) || x_0 - p || + \gamma_n a || x_n - p || \)
\( \leq 0 \leq 1 \) (5)

Therefore, taking the limit as \( n \to \infty \) of both sides of the inequality (5) and using lemma 1.6 we get
\( \lim_{n \to \infty} ||x_n - p|| = 0 \), That is \( \lim_{n \to \infty} x_n = x^* \)

By theorem 3.2 \( \| x_n - p \| \leq \| x_{n+1} - p \| \)