

Research Article

Stability of the Generalized Polar Decomposition Method for the Approximation of the Matrix Exponential

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Abstract Generalized polar decomposition method (or briefly GPD method) has been introduced by Munthe-Kaas and Zanna [5] to approximate the matrix exponential. In this paper, we investigate the numerical stability of that method with respect to roundoff propagation. The numerical GPD method includes two parts: splitting of a matrix $Z \in \mathfrak{g}$, a Lie algebra of matrices and computing $\exp(Z)\mathbf{v}$ for a vector \mathbf{v} . We show that the former is stable provided that $\|Z\|$ is not so large, while the latter is not stable in general except with some restrictions on the entries of the matrix Z and the vector \mathbf{v} .

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1 Introduction

Since exponential matrix naturally appears in the analytical solutions of many differential equations, it is important to present the numerical approximate methods for computing exponential matrices in applied problems. In many cases that the different equations are modeled on a matrix Lie group, we need that the approximated matrix to remain in the same Lie group. Most of the well-known standard methods [3] do not satisfy this condition except in some special cases, like 3×3 antisymmetric matrices [2].

Recently, Munthe-Kaas and Zanna have introduced a geometric numerical method, called generalized polar decomposition method, or briefly GPD method [5]. The main advantage of this method among others is that it preserves the approximated object in the mentioned Lie group. In the GPD method, the Lie group G and its Lie algebra \mathfrak{g} are splitted to simpler ones, such that the computation of $\exp Z$ will be easier in those subgroups and subalgebras. There are two numerical algorithms in GPD method. The first one (Algorithm 1) splits the matrix Z and in the second one (Algorithm 2), $\exp(Z)\mathbf{v}$ is computed, where \mathbf{v} is an arbitrary vector.

As far as we know, no proof of the stability of the GPD method, with respect to propagation of roundoff error, has appeared in the literature. In this paper, we study the stability of the method for the so-called *polar-type splitting order-two* algorithm that has been introduced in [5]. We show that Algorithm 1 which is a simpler algorithm is stable, provided the norm $\|Z\| = \max_{i,j} |Z_{i,j}|$ of the matrix Z is not very large, but in the complicated one, that is, Algorithm 2, the stability is controlled by entries of Z , $\exp \|Z\|$ and $\|\mathbf{v}\| = \max_i |v_i|$ as the norm of the vector \mathbf{v} .

We introduce a sufficient condition for this aim. More precisely, given an $n \times n$ matrix Z , let

$$\mathbf{a}_j = [Z_{j+1,j}, Z_{j+2,j}, \dots, Z_{n,j}], \quad \mathbf{b}_j = [Z_{j,j+1}, Z_{j,j+2}, \dots, Z_{j,n}]^T$$

for $j = 1, 2, \dots, n-1$ and let

$$k_0 = \min \{ |\mathbf{b}_j^T \mathbf{a}_j|; \mathbf{b}_j^T \mathbf{a}_j \neq 0, 1 \leq j \leq n-1 \},$$

then Algorithm 2 is stable under the following conditions:

$$\|Z\| \text{ and } \|\mathbf{v}\| \text{ are not very large,} \tag{1.1a}$$

$$\frac{\|Z\|}{k_0} < \left(\frac{1-u}{1+u} \right) \left(\frac{1-nu}{n^2u} \right), \tag{1.1b}$$

where u is the unit roundoff error.

The content of the paper is as follows. In Section 2, some preliminaries concerning the general concept of error analysis and GPD method and also details of the above-mentioned algorithms are given. In Section 3, the error analysis of the Algorithm 1 and in Section 4, the error analysis of Algorithm 2 are addressed. Section 5 is devoted to sensibility analysis of Algorithm 2.

2 Preliminaries

2.1 Error analysis

In this section, we recall some standard definitions and lemmas, which are applied in the other sections. Elementary operators like $+$, \cdot , \times , and $/$ as well as $\sqrt{\cdot}$, \sin , and \cos admit the following floating point arithmetic:

$$\mathfrak{fl}(x \text{ op } y) = (x \text{ op } y)(1 + \delta), \quad |\delta| \leq u,$$

where op is an elementary operation, and u is the unit roundoff error.

Lemma 1 (see [1]). *If $|\delta_i| \leq u$ and $\rho_i = \pm 1$ for $1 \leq i \leq n$ and $nu < 1$, then*

$$\prod_{i=1}^n (1 + \delta_i)^{\rho_i} = 1 + \theta_n,$$

such that

$$|\theta_n| \leq \frac{nu}{1 - nu} := \gamma_n.$$

The following technical properties are satisfied by the sequence $\{\gamma_n\}_n$:

$$\gamma_j + \gamma_k + \gamma_j \gamma_k \leq \gamma_{j+k}, \quad (j, k \geq 1), \quad (2.1)$$

$$c\gamma_n \leq \gamma_{cn}, \quad (c \geq 1, n > 1). \quad (2.2)$$

Denoting the calculated value of any variable x by \hat{x} and $|A| = [|a_{ij}|]$ for any matrix $A = [a_{ij}]$, we have the following lemma.

Lemma 2 (see [1,4]). *If $y = (c - \sum_{i=1}^{k-1} a_i b_i) / b_k$ is evaluated in floating point arithmetic, then, no matter what is the order of evaluation, there exist constants $\theta_k^{(i)}$, $i = 0, 1, \dots, k-1$, such that*

$$b_k \hat{y} (1 + \theta_k^{(0)}) = c - \sum_{i=1}^{k-1} a_i b_i (1 + \theta_k^{(i)}), \quad (2.3)$$

where $|\theta_k^{(i)}| \leq \gamma_k$ for all i . If $b_k = 1$, so that there is no division, then $|\theta_k^{(i)}| \leq \gamma_{k-1}$ for all i .

Also for matrix multiplication, we have

$$\mathfrak{fl}(AB) = AB + \Delta, \quad |\Delta| \leq \gamma_n |A| |B|, \quad A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times p}. \quad (2.4)$$

For any matrix $A = [a_{ij}]$, let $\|A\| = \max_{i,j} |a_{ij}|$. By this norm, we have

$$\|AB\| \leq n \|A\| \|B\| \quad (2.5)$$

for any $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$.

For any two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, let

$$\mathbf{a} * \mathbf{b} = (a_1 b_1, \dots, a_n b_n) \in \mathbb{R}^n. \quad (2.6)$$

So, using this notation, we can write

$$\mathfrak{fl}(\lambda \mathbf{a}) = \lambda \mathbf{a} * (\mathbf{1} + \mathbf{u}), \quad |\mathbf{u}| \leq u \mathbf{1}, \quad \mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n, \lambda \in \mathbb{R}, \quad (2.7)$$

Similar to Lemma 1, one can prove the following lemma.

Lemma 3. *If $|\mathbf{u}_i| \leq u \mathbf{1}$, for $1 \leq i \leq n$ and $nu < 1$, then*

$$(\mathbf{1} + \mathbf{u}_1) * \dots * (\mathbf{1} + \mathbf{u}_n) = \mathbf{1} + \mathbf{h}_n, \quad (2.8)$$

such that

$$|\mathbf{h}_n| \leq \gamma_n \mathbf{1}.$$

2.2 GPD method

Suppose G is a finite dimensional Lie group and \mathfrak{g} is its Lie algebra and let $\sigma : G \rightarrow G$, $\sigma \neq id$ be an involutive automorphism, that is, one-to-one smooth map such that:

$$\begin{aligned}\sigma(x \cdot y) &= \sigma(x) \cdot \sigma(y), \quad \forall x, y \in G, \\ \sigma(\sigma(x)) &= x, \quad \forall x \in G.\end{aligned}$$

It is well known that for sufficiently small t and for any $z = \exp(tZ) \in G$, where $Z \in \mathfrak{g}$, we can write

$$z = xy, \tag{2.9}$$

where $\sigma(x) = x^{-1}$ and $\sigma(y) = y$. The decomposition (2.9) is called the *Generalized Polar Decomposition* (GPD) of z . On the Lie algebra \mathfrak{g} , the involutive automorphism is induced by σ :

$$d\sigma(Z) = \left. \frac{d}{dt} \right|_{t=0} \sigma(\exp tZ),$$

hence $d\sigma$ defines a splitting of \mathfrak{g} into the direct sum of two linear spaces:

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k},$$

where $\mathfrak{k} = \{Z \in \mathfrak{g} : d\sigma(Z) = Z\}$ and $\mathfrak{p} = \{Z \in \mathfrak{g} : d\sigma(Z) = -Z\}$. In fact, \mathfrak{k} is a Lie subalgebra of \mathfrak{g} , but \mathfrak{p} only admits Lie triple system property:

$$A, B, C \in \mathfrak{p} \implies [A, [B, C]] \in \mathfrak{p},$$

where $[A, B]$ is the standard commutator on Lie algebra \mathfrak{g} . Now, let

$$P = \Pi_{\mathfrak{p}}(Z), \quad K = \Pi_{\mathfrak{k}}(Z),$$

where the maps $\Pi_{\mathfrak{p}} : \mathfrak{g} \rightarrow \mathfrak{p}$ and $\Pi_{\mathfrak{k}} : \mathfrak{g} \rightarrow \mathfrak{k}$ are the canonical projection maps. It can be easily verified that every $Z \in \mathfrak{g}$ can be uniquely decomposed into $Z = P + K$, where

$$P = \Pi_{\mathfrak{p}}(Z) = \frac{1}{2}(Z - d\sigma(Z)), \quad K = \Pi_{\mathfrak{k}}(Z) = \frac{1}{2}(Z + d\sigma(Z)),$$

(see [5]). Also, \mathfrak{k} and \mathfrak{p} satisfy

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}], [\mathfrak{p}, \mathfrak{k}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}.$$

If, in the decomposition (2.9), we consider $x = \exp X(t)$ and $y = \exp Y(t)$, where $X(t) \in \mathfrak{p}$ and $Y(t) \in \mathfrak{k}$, for sufficiently small t , then it can be written as follows:

$$X(t) = \sum_{i=1}^{\infty} X_i t^i, \quad Y(t) = \sum_{i=1}^{\infty} Y_i t^i,$$

where the coefficients X_i and Y_i can be found in [5]. The first terms in the expansions of $X(t)$ and $Y(t)$ are

$$\begin{aligned}X &= Pt - \frac{1}{2}[P, K]t^2 - \frac{1}{6}[K, [P, K]]t^3 + \left(\frac{1}{24}[P, [P, [P, K]]] - \frac{1}{24}[K, [K, [P, K]]] \right)t^4 \\ &+ \left(\frac{7}{360}[K, [P, [P, [P, K]]]] - \frac{1}{120}[K, [K, [K, [P, K]]]] - \frac{1}{180}[[P, K], [P, [P, K]]] \right)t^5 + O(t^6),\end{aligned} \tag{2.10}$$

$$\begin{aligned}Y &= Kt - \frac{1}{12}[P, [P, K]]t^3 \\ &+ \left(\frac{1}{120}[P, [P, [P, [P, K]]]] + \frac{1}{720}[K, [K, [P, [P, K]]]] - \frac{1}{240}[[P, K], [K, [P, K]]] \right)t^5 + O(t^7).\end{aligned} \tag{2.11}$$

Considering $\sigma_1, \sigma_2, \dots, \sigma_m$ as elements of a sequence of involutive automorphisms on G , σ_1 induces a decomposition $\mathfrak{g} = \mathfrak{p}_1 \oplus \mathfrak{k}_1$ and approximation:

$$\exp(tZ) \approx \exp(X^{[1]}(t)) \exp(Y^{[1]}(t)),$$

where $X^{[1]}$ and $Y^{[1]}$ are suitable truncations of (2.10) and (2.11), respectively. Then \mathfrak{k}_2 can be decomposed by σ_2 and so we have $\mathfrak{k}_1 = \mathfrak{p}_2 \oplus \mathfrak{k}_2$ and

$$\exp(Y^{[1]}(t)) \approx \exp(X^{[2]}(t)) \exp(Y^{[2]}(t)).$$

Repeating this process m times, ($m \geq 1$) yields the following:

$$\mathfrak{g} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \cdots \oplus \mathfrak{p}_m \oplus \mathfrak{k}_m, \quad \exp(tZ) \approx \exp(X^{[1]}(t)) \cdots \exp(X^{[m]}(t)) \exp(Y^{[m]}(t)). \quad (2.12)$$

The number m is chosen appropriately such that the right-hand side terms can be easily computed. If the expansions (2.10) are truncated at order two, then the two following algorithms based on the iterated generalized polar decomposition (2.12) will be obtained. The first algorithm splits the matrix Z , while the second one approximates $\exp(Z)\mathbf{v}$.

Algorithm 1 (Polar-type splitting, order two).

% Purpose: 2nd order approximation of the splitting (2.12)

% In: $n \times n$ matrix Z

% Out: Z overwritten with the nonzero elements of $X^{[i]}$ and $Y^{[m]}$ as:

% $Z(i+1:n, i) = X^{[i]}(i+1:n, i)$, $Z(i, i+1:n) = X^{[i]}(i, i+1:n)$

% $\text{diag}(Z) = \text{diag}(Y^{[m]})$

for $j = 1 : n - 1$

$\mathbf{a}_j := Z(j+1:n, j)$

$\mathbf{b}_j := Z(j, j+1:n)^T$

$\overline{K}_j := Z(j+1:n, j+1:n)$

$\mathbf{c}_j := z_{j,j} \mathbf{a}_j - \overline{K}_j \mathbf{a}_j$

$\mathbf{d}_j := -z_{j,j} \mathbf{b}_j + \overline{K}_j^T \mathbf{b}_j$

$Z(j+1:n, j) := \mathbf{a}_j - \frac{1}{2} \mathbf{c}_j$

$Z(j, j+1:n) := (\mathbf{b}_j - \frac{1}{2} \mathbf{d}_j)^T$

end

Algorithm 2 (Polar-type approximation).

% Purpose: Computing the approximant (2.12) applied to a vector \mathbf{v}

% In: \mathbf{v} : n -vector

% Z : $n \times n$ matrix containing the nonzero elements of $X^{[i]}$ and $Y^{[m]}$ as:

% $Z(i+1:n, i) = X^{[i]}(i+1:n, i)$, $Z(i, i+1:n) = X^{[i]}(i, i+1:n)$

% $\text{diag}(Z) = \text{diag}(Y^{[m]})$

% Out: $\mathbf{v} := \exp(X^{[1]}) \cdots \exp(X^{[m]}) \exp(Y^{[m]}) \mathbf{v}$

for $k = 1 : n$

$v_k := \exp(z_{k,k}) v_k$

end

for $j = n - 1 : -1 : 1$

$\mathbf{a}_j := [0; Z(j+1:n, j)]$

$\mathbf{b}_j := [0; Z(j, j+1:n)^T]$

$\mathbf{v}_{\text{old}} := \mathbf{v}(j:n)$

$\alpha_j := \sqrt{\mathbf{b}_j^T \mathbf{a}_j}$, $\beta_j = \frac{\sinh \alpha_j}{\alpha_j}$, $\lambda_j = 2 \sinh^2(\frac{\alpha_j}{2})$

$D := \begin{pmatrix} 0 & 1 \\ \alpha_j^2 & 0 \end{pmatrix}$

$\varphi(D) := \lambda_j D^{-1} + \beta_j I$,

$\mathbf{w} := \varphi(D) \begin{pmatrix} v_{\text{old},1} \\ \mathbf{b}_j^T \mathbf{v}_{\text{old}} \end{pmatrix}$

$\mathbf{v}_{\text{new}} := [\mathbf{a}_j, \mathbf{e}_1] \mathbf{w} = w_1 \mathbf{a}_j + w_2 \mathbf{e}_1$

$\mathbf{v}(j:n) = \mathbf{v}_{\text{old}} + \mathbf{v}_{\text{new}}$

end

3 Error analysis of Algorithm 1

In Algorithm 1, we denote by Z simultaneously the input data matrix and the splitted output matrix. In this algorithm, the calculated value of \mathbf{c}_j ($1 \leq j \leq n-1$) is

$$\widehat{\mathbf{c}}_j = (z_{j,j} \mathbf{a}_j * (\mathbf{1} + \mathbf{u}_j^1) - \overline{K}_j \mathbf{a}_j - \mathbf{u}_j^2) * (\mathbf{1} + \mathbf{u}_j^3),$$

where $|\mathbf{u}_j^i| \leq u\mathbf{1}$ ($i = 1, 3$) and $|\mathbf{u}_j^2| \leq \gamma_{n-j} |\overline{K}_j| |\mathbf{a}_j|$. From (2.8), we have $(\mathbf{1} + \mathbf{u}_j^1) * (\mathbf{1} + \mathbf{u}_j^3) = \mathbf{1} + \mathbf{h}_2$, where $|\mathbf{h}_2| \leq \gamma_2 \mathbf{1}$.

Hence,

$$\widehat{\mathbf{c}}_j = \mathbf{c}_j + \mathbf{u}_j,$$

where

$$\mathbf{u}_j = z_{j,j} \mathbf{a}_j * \mathbf{h}_2 - \overline{K}_j \mathbf{a}_j * \mathbf{u}_j^3 - \mathbf{u}_j^2 * (\mathbf{1} + \mathbf{u}_j^3), \quad (3.1)$$

$$|\mathbf{u}_j| \leq \gamma_2 |z_{j,j}| |\mathbf{a}_j| + u |\overline{K}_j| |\mathbf{a}_j| + \gamma_{n-j} |\overline{K}_j| |\mathbf{a}_j| * (\mathbf{1} + u\mathbf{1}). \quad (3.2)$$

By applying the norm of the matrix Z this reduces to

$$|\mathbf{u}_j| \leq ((\gamma_2 + u + \gamma_{n-j}(1 + u)) \|Z\|^2) \mathbf{1}.$$

Now, similarly for computing \mathbf{d}_j , we have

$$\widehat{\mathbf{d}}_j = \mathbf{d}_j + \mathbf{u}'_j,$$

where

$$|\mathbf{u}'_j| \leq ((\gamma_2 + u + \gamma_{n-j}(1 + u)) \|Z\|^2) \mathbf{1}.$$

Therefore, the entries of the matrix Z change as follows:

$$\text{fl} \left(\mathbf{a}_j - \frac{1}{2} \mathbf{c}_j \right) = \left(\mathbf{a}_j - \frac{1}{2} (\mathbf{c}_j + \mathbf{u}_j) * (\mathbf{1} + \mathbf{u}_j^4) \right) * (\mathbf{1} + \mathbf{u}_j^5) = \left(\mathbf{a}_j - \frac{1}{2} \mathbf{c}_j \right) + \mathbf{p}_j,$$

where $|\mathbf{u}_j^i| \leq u\mathbf{1}$, ($i = 4, 5$) and

$$\mathbf{p}_j = \mathbf{a}_j * \mathbf{u}_j^5 - \frac{1}{2} \mathbf{c}_j * \mathbf{h}_2 - \frac{1}{2} \mathbf{u}_j * (\mathbf{1} + \mathbf{h}_2).$$

So from (3.1),

$$|\mathbf{p}_j| \leq u |\mathbf{a}_j| + \frac{1}{2} |\mathbf{c}_j| \gamma_2 + \frac{1}{2} (1 + \gamma_2) (\gamma_2 |z_{j,j}| |\mathbf{a}_j| + (u + \gamma_{n-j}(1 + u)) |\overline{K}_j| |\mathbf{a}_j|).$$

If we replace the value of \mathbf{c}_j from Algorithm 1 in the above relation, we will have

$$|\mathbf{p}_j| \leq u |\mathbf{a}_j| + \left(\gamma_2 + \frac{1}{2} \gamma_2^2 \right) |z_{j,j}| |\mathbf{a}_j| + \left(\frac{1}{2} \gamma_2 + \frac{1}{2} (1 + \gamma_2) (u + \gamma_{n-j}(1 + u)) \right) |\overline{K}_j| |\mathbf{a}_j|,$$

and finally

$$|\mathbf{p}_j| \leq \left(u \|Z\| + \left(\gamma_2 + \frac{1}{2} \gamma_2^2 + \frac{1}{2} \gamma_2 + \frac{1}{2} (1 + \gamma_2) (u + \gamma_{n-j}(1 + u)) \right) \|Z\|^2 \right) \mathbf{1}.$$

Since

$$\gamma_{n-j} = \frac{(n-j)u}{1 - (n-j)u} < \frac{(n-j)u}{1 - nu} = \frac{n-j}{n} \gamma_n$$

for $nu < 1$ and from property (2.1), the upper bound of \mathbf{p}_j will be

$$|\mathbf{p}_j| \leq \left(\|Z\|u + \left(\frac{2}{1-2u} + \frac{n-j+2}{2(1-nu)} + \frac{1}{2} \right) \|Z\|^2 u + O(u^2) \right) \mathbf{1}.$$

The greatest value of the right-hand side occurs at $j = 1$, hence for all j ,

$$|\mathbf{p}_j| \leq \left(\|Z\|u + \left(\frac{2}{1-2u} + \frac{n+1}{2(1-nu)} + \frac{1}{2} \right) \|Z\|^2u + O(u^2) \right) \mathbf{1}. \quad (3.3)$$

Similarly, if

$$\text{fl} \left(\mathbf{b}_j - \frac{1}{2} \mathbf{d}_j \right) = \left(\mathbf{b}_j - \frac{1}{2} \mathbf{d}_j \right) + \mathbf{q}_j,$$

then

$$|\mathbf{q}_j| \leq \left(\|Z\|u + \left(\frac{2}{1-2u} + \frac{n+1}{2(1-nu)} + \frac{1}{2} \right) \|Z\|^2u + O(u^2) \right) \mathbf{1}. \quad (3.4)$$

However, since the diagonal entries do not change during this algorithm, (3.3) and (3.4) show the stability of the algorithm whenever $\|Z\|$ is not so large.

4 Error analysis of Algorithm 2

In the GPD method, the matrix Z is splitted in Algorithm 1, and the $\exp(Z)\mathbf{v}$ is calculated in Algorithm 2. We denote the splitted matrix of Z with $\bar{Z} = (\bar{z}_{i,j})$.

4.1 Error analysis of α_j

Let $k_j = \mathbf{b}_j^T \mathbf{a}_j$ and $\alpha_j = \sqrt{k_j}$, here we have assumed $k_j > 0$. If k_j is not a positive number, by considering $\alpha_j = \sqrt{|k_j|}$ when $k_j < 0$ and by letting $\frac{\sinh \alpha_j}{\alpha_j} = 1$ when $k_j = 0$, it is easily seen that of all the arguments and the upper bounds presented in the paper remain valid.

From Lemma 2, we have

$$\text{fl}(k_j) = \sum_{i=j+1}^n \bar{z}_{j,i} \bar{z}_{i,j} \left(1 + \theta_{n-j}^{(i)} \right),$$

where $|\theta_{n-j}^{(i)}| \leq \gamma_{n-j}$, ($1 \leq i \leq n$). So we can write

$$\text{fl}(k_j) = k_j + \sum_{i=j+1}^n \bar{z}_{j,i} \bar{z}_{i,j} \theta_{n-j}^{(i)} = k_j + t,$$

where

$$|t| \leq (n-j)\gamma_{n-j} \|\bar{Z}\|^2. \quad (4.1)$$

Then by assuming $\|Z\| \leq 1$ and from condition (1.1), we have

$$|t| \leq n\gamma_n \leq \frac{1-u}{1+u} k_0 \leq k_0,$$

where $k_0 = \min\{|\mathbf{b}_j^T \mathbf{a}_j|; \mathbf{b}_j^T \mathbf{a}_j \neq 0, 1 \leq j \leq n-1\}$. Hence $k_j + t$ is nonnegative and

$$\text{fl}(\alpha_j) = \sqrt{k_j + t(1 + \delta)},$$

where δ , with $|\delta| \leq u$, denotes the floating point arithmetic error. Therefore,

$$\text{fl}(\alpha_j) = \alpha_j + \xi_j, \quad (4.2)$$

where

$$\xi_j = \alpha_j \delta + \left(\frac{t}{\sqrt{k_j + t} + \alpha_j} \right) (1 + \delta).$$

Since $\alpha_j \leq \sqrt{n-j} \|\bar{Z}\|$, we have

$$|\xi_j| \leq \alpha_j u + \left(\frac{|t|}{\alpha_j} \right) (1 + u) \leq \alpha_j u + \left(\frac{(n-j)\gamma_{n-j} \|\bar{Z}\|^2}{\alpha_j} \right) (1 + u) \leq \varphi_j \|\bar{Z}\|, \quad (4.3)$$

where

$$\varphi_j := u\sqrt{n-j} + \frac{(n-j)\gamma_{n-j} \|\bar{Z}\|}{\sqrt{k_0}} (1 + u). \quad (4.4)$$

4.2 Error analysis of β_j

Let us consider

$$\text{fl}(\beta_j) = \frac{\sinh(\alpha_j + \xi_j)}{\alpha_j + \xi_j}(1 + \delta).$$

By using the identity $\sinh x - \sinh y = 2 \sinh \frac{x-y}{2} \cosh \frac{x+y}{2}$, we obtain

$$\text{fl}(\beta_j) = \left(\beta_j + \frac{\xi_j \frac{\sinh \xi_j/2}{\xi_j/2} \cosh(\alpha_j + \xi_j/2) - \beta_j \xi_j}{\alpha_j + \xi_j} \right) (1 + \delta) = \beta_j + \varepsilon_j^1,$$

where the quantity ε_j^1 is

$$\varepsilon_j^1 = \beta_j \delta + \left(\frac{\xi_j \frac{\sinh \xi_j/2}{\xi_j/2} \cosh(\alpha_j + \xi_j/2) - \beta_j \xi_j}{\alpha_j + \xi_j} \right) (1 + \delta). \quad (4.5)$$

From (4.1) and (4.3), we have

$$|\xi_j/\alpha_j| \leq u + \frac{(n-j)\gamma_{n-j}}{k_0} \|\bar{Z}\|^2 (1+u). \quad (4.6)$$

We now consider the condition

$$\psi_j := u + \frac{(n-j)\gamma_{n-j}}{k_0} \|\bar{Z}\|^2 (1+u) < 1 \quad (4.7)$$

that is clearly a consequence of

$$\frac{\|\bar{Z}\|^2}{k_0} < \left(\frac{1-u}{1+u} \right) \left(\frac{1-nu}{n^2u} \right)$$

for $1 \leq j \leq n-1$, which itself is obtained from condition (1.1b) for $\|\bar{Z}\| < 1$. Hence from (4.7) and by increasing monotonicity of the map $x \mapsto \frac{x}{1-x}$ on the interval $(0, 1)$, we can write

$$\left| \frac{\xi_j}{\alpha_j + \xi_j} \right| \leq \frac{|\xi_j/\alpha_j|}{1 - |\xi_j/\alpha_j|} \leq \frac{\psi_j}{1 - \psi_j}.$$

The function

$$f(x) = \begin{cases} \frac{\sinh x}{x}, & x \neq 0, \\ 1, & x = 0 \end{cases}$$

has the elementary property that $1 \leq f(a) \leq f(b)$ whenever $|a| \leq b$, from which and (4.5) we deduce $|\varepsilon_j^1| \leq \eta_j$, where

$$\begin{aligned} \eta_j &:= \frac{\sinh \sqrt{n-j} \|\bar{Z}\|}{\sqrt{n-j} \|\bar{Z}\|} \left(u + \frac{\psi_j}{1 - \psi_j} (1+u) \right) \\ &+ \frac{\psi_j}{1 - \psi_j} \frac{\sinh \varphi_j \|\bar{Z}\|/2}{\varphi_j \|\bar{Z}\|/2} \cosh \left(\sqrt{n-j} \|\bar{Z}\| + \varphi_j \|\bar{Z}\|/2 \right) (1+u). \end{aligned} \quad (4.8)$$

Now by using (4.4) and (4.7) in (4.8), we get

$$\begin{aligned} |\varepsilon_j^1| &\leq \frac{\sinh(\sqrt{n-j} \|\bar{Z}\|)}{\sqrt{n-j} \|\bar{Z}\|} \left(\frac{2u + (n-j)\gamma_{n-j} \frac{\|\bar{Z}\|^2}{k_0} (1+u)}{1 - u - (n-j)\gamma_{n-j} \frac{\|\bar{Z}\|^2}{k_0} (1+u)} \right) \\ &+ \frac{\sinh \left(u\sqrt{n-j} \|\bar{Z}\|/2 + (n-j)\gamma_{n-j} \frac{\|\bar{Z}\|^2}{2\sqrt{k_0}} (1+u) \right)}{u\sqrt{n-j} \|\bar{Z}\|/2 + (n-j)\gamma_{n-j} \frac{\|\bar{Z}\|^2}{2\sqrt{k_0}} (1+u)} \\ &\times \cosh \left(\left(1 + \frac{u}{2} \right) \sqrt{n-j} \|\bar{Z}\| + (n-j)\gamma_{n-j} \frac{\|\bar{Z}\|^2}{2\sqrt{k_0}} (1+u) \right) \\ &\times \frac{u + (n-j)\gamma_{n-j} \frac{\|\bar{Z}\|^2}{k_0} (1+2u)}{1 - u - (n-j)\gamma_{n-j} \frac{\|\bar{Z}\|^2}{k_0} (1+u)} + O(u^2). \end{aligned}$$

This shows that $\varepsilon_j^1 \rightarrow 0$ as $\|\bar{Z}\| \rightarrow 0$, provided that $\frac{\|\bar{Z}\|}{k_0}$ admits an upper bound as suggested by the condition (1.1).

4.3 Error analysis of λ_j

However, for computing λ_j , from (4.2) and definition of floating point arithmetic error δ we have

$$\sinh\left(\left(\frac{\alpha_j + \xi_j}{2}\right)(1 + \delta)\right) = \sinh\frac{\alpha_j}{2}\left(\cosh\frac{\alpha_j\delta + \xi_j(1 + \delta)}{2} + \coth\frac{\alpha_j}{2}\sinh\frac{\alpha_j\delta + \xi_j(1 + \delta)}{2}\right) = \sinh\frac{\alpha_j}{2}(1 + \varepsilon_j^2),$$

where

$$\varepsilon_j^2 = 2\sinh^2\frac{\alpha_j\delta + \xi_j(1 + \delta)}{4} + \coth\frac{\alpha_j}{2}\sinh\frac{\alpha_j\delta + \xi_j(1 + \delta)}{2}.$$

Hence,

$$\text{fl}(\lambda_j) = 2\sinh^2\frac{\alpha_j}{2}(1 + \varepsilon_j^2)^2(1 + \theta_3).$$

So,

$$\text{fl}(\lambda_j) = \lambda_j(1 + \varepsilon_j^3),$$

where

$$\varepsilon_j^3 = \varepsilon_j^2(\varepsilon_j^2 + 2)(1 + \theta_3) + \theta_3. \quad (4.9)$$

Also, $|\varepsilon_j^2| \leq \rho_j$, where

$$\begin{aligned} \rho_j &:= 2\sinh^2\frac{\sqrt{n-j}\|\bar{Z}\|u + \varphi_j(1+u)\|\bar{Z}\|}{4} \\ &\quad + \coth\frac{\sqrt{k_0}}{2}\sinh\frac{\sqrt{n-j}\|\bar{Z}\|u + \varphi_j(1+u)\|\bar{Z}\|}{2}, \\ |\rho_j| &\leq 2\sinh^2\left(\frac{1}{2}u\sqrt{n-j}\|\bar{Z}\| + \frac{(n-j)\gamma_{n-j}\|\bar{Z}\|^2}{4\sqrt{k_0}}(1+2u)\right) \\ &\quad + \coth\left(\frac{\sqrt{k_0}}{2}\right)\sinh\left(u\sqrt{n-j}\|\bar{Z}\| + \frac{(n-j)\gamma_{n-j}\|\bar{Z}\|^2}{2\sqrt{k_0}}(1+2u)\right) + O(u^2). \end{aligned}$$

However,

$$\coth\frac{\sqrt{k_0}}{2} = \frac{\cosh\frac{\sqrt{k_0}}{2}}{\sinh\frac{\sqrt{k_0}}{2}} \leq \frac{\cosh\frac{\sqrt{k_0}}{2}}{\frac{\sqrt{k_0}}{2}},$$

and therefore

$$\begin{aligned} |\rho_j| &\leq 2\sinh^2\left(\frac{1}{2}u\sqrt{n-j}\|\bar{Z}\| + \frac{(n-j)\gamma_{n-j}\|\bar{Z}\|^2}{4\sqrt{k_0}}(1+2u)\right) \\ &\quad + \cosh\left(\frac{\sqrt{k_0}}{2}\right) \times \frac{\sinh\left(u\sqrt{n-j}\|\bar{Z}\| + \frac{(n-j)\gamma_{n-j}\|\bar{Z}\|^2}{2\sqrt{k_0}}(1+2u)\right)}{u\sqrt{n-j}\|\bar{Z}\| + \frac{(n-j)\gamma_{n-j}\|\bar{Z}\|^2}{2\sqrt{k_0}}(1+2u)} \\ &\quad \times \left(2u\sqrt{n-j}\frac{\|\bar{Z}\|}{\sqrt{k_0}} + \frac{(n-j)\gamma_{n-j}\|\bar{Z}\|^2}{k_0}(1+2u)\right) + O(u^2). \end{aligned}$$

Hence in this case $\varepsilon_j^2 \rightarrow 0$ as $\|\bar{Z}\| \rightarrow 0$, provided that $\frac{\|\bar{Z}\|}{k_0}$ admits an upper bound like the one suggested by condition (1.1) (note that $\frac{\|\bar{Z}\|}{\sqrt{k_0}} = \sqrt{\frac{\|\bar{Z}\|}{k_0}} \cdot \sqrt{\|\bar{Z}\|}$). Moreover, by (4.9),

$$|\varepsilon_j^3| \leq o_j, \quad (4.10)$$

where

$$o_j := (\rho_j^2 + 2\rho_j)(1 + \gamma_3) + \gamma_3, \quad (4.11)$$

which implies o_j is of order $O(u)$ under condition (1.1).

4.4 Error analysis of $\varphi(D)$

Now, in calculating $\varphi(D)$, we will have

$$\text{fl}(\varphi(D)) = \begin{pmatrix} \text{fl}(\beta_j) & \text{fl}(\lambda_j) \\ \text{fl}(\lambda_j) & \text{fl}(\beta_j) \end{pmatrix} \begin{pmatrix} \frac{\text{fl}(\lambda_j)}{\text{fl}(\alpha_j^2)}(1+\delta) \\ (1+\delta) \end{pmatrix} = \begin{pmatrix} \beta_j(1+\varepsilon_j^1) & \frac{\lambda_j(1+\varepsilon_j^3)}{(\alpha_j+\xi_j)^2}(1+\delta) \\ \lambda_j(1+\varepsilon_j^j) & \beta_j(1+\varepsilon_j^1) \end{pmatrix} = \varphi(D) + A,$$

where

$$A = \begin{pmatrix} \beta_j \varepsilon_j^1 & \frac{\lambda_j}{\alpha_j^2} \sigma_j \\ \lambda_j \varepsilon_j^3 & \beta_j \varepsilon_j^1 \end{pmatrix}, \quad \sigma_j = \frac{(1+\varepsilon_j^3)(1+\delta)}{(1+\xi_j/\alpha_j)^2} - 1.$$

From (4.6) and (4.10), we have

$$|\sigma_j| \leq \frac{o_j(1+u) + u + 2\psi_j + \psi_j^2}{(1-\psi_j)^2}. \tag{4.12}$$

Hence by (4.7) and (4.11), the right-hand side of (4.12) is of order $O(u)$ and

$$|\varphi(D)| \leq B, \tag{4.13}$$

where

$$B := \begin{pmatrix} \frac{\sinh \sqrt{n-j}\|\bar{Z}\|}{\sqrt{n-j}\|\bar{Z}\|} & \frac{2 \sinh^2(\sqrt{n-j}\|\bar{Z}\|/2)}{(n-j)\|\bar{Z}\|^2} \\ 2 \sinh^2(\sqrt{n-j}\|\bar{Z}\|/2) & \frac{\sinh \sqrt{n-j}\|\bar{Z}\|}{\sqrt{n-j}\|\bar{Z}\|} \end{pmatrix}, \tag{4.14}$$

$$|A| \leq C, \tag{4.15}$$

$$C := \begin{pmatrix} \frac{\sinh \sqrt{n-j}\|\bar{Z}\|}{\sqrt{n-j}\|\bar{Z}\|} \eta_j & \frac{2 \sinh^2(\sqrt{n-j}\|\bar{Z}\|/2)}{(n-j)\|\bar{Z}\|^2} |\sigma_j| \\ 2 \sinh^2(\sqrt{n-j}\|\bar{Z}\|/2) o_j & \frac{\sinh \sqrt{n-j}\|\bar{Z}\|}{\sqrt{n-j}\|\bar{Z}\|} \eta_j \end{pmatrix}. \tag{4.16}$$

4.5 Error analysis of \mathbf{w}

At first, since $v_k := \exp(z_{k,k})v_k$ in Algorithm 2, we have $\hat{v}_{\text{old},i} = v_{\text{old},i}(1+\theta_2^{(i)})$, $(1 \leq i \leq n)$. Now, for computing \mathbf{w} , let $v' = \mathbf{b}_j^T \mathbf{v}_{\text{old}}$, hence,

$$\text{fl}(v') = \sum_{i=j+1}^n \bar{z}_{j,i} v_{\text{old},i} (1+\theta_2^{(i)}) (1+\theta_{n-j}^{(i)}) = v' + \varepsilon_j^4,$$

where

$$\varepsilon_j^4 := \sum_{i=j+1}^n \bar{z}_{j,i} v_{\text{old},i} (\theta_2^{(i)} + \theta_{n-j}^{(i)} + \theta_2^{(i)} \theta_{n-j}^{(i)}).$$

So from property (2.1), we have

$$|\varepsilon_j^4| \leq (n-j)\gamma_{n-j+2}\|\bar{Z}\| \|\mathbf{v}\| e^{\|\bar{Z}\|}. \tag{4.17}$$

Now from Algorithm 2 and (2.4),

$$\text{fl}(\mathbf{w}) = (\varphi(D) + A) \begin{pmatrix} v_{\text{old},1}(1+\theta_2) \\ v' + \varepsilon_j^4 \end{pmatrix} + \mathbf{u}_j = \mathbf{w} + \mathbf{q}_j,$$

where

$$\mathbf{q}_j := \varphi(D) \begin{pmatrix} v_{\text{old},1}\theta_2 \\ \varepsilon_j^4 \end{pmatrix} + A \begin{pmatrix} v_{\text{old},1}(1+\theta_2) \\ v' + \varepsilon_j^4 \end{pmatrix} + \mathbf{u}_j, \quad (4.18)$$

$$|\mathbf{u}_j| \leq \gamma_2 |\varphi(D) + A| \begin{pmatrix} |v_{\text{old},1}(1+\theta_2)| \\ |v' + \varepsilon_j^4| \end{pmatrix}. \quad (4.19)$$

Therefore, from (4.13), (4.15), (4.17), (4.18), and (4.19), we have $|\mathbf{q}_j| \leq \mathbf{r}_j$, where

$$\begin{aligned} \mathbf{r}_j := & \|\mathbf{v}\| e^{\|\bar{Z}\|} B \begin{pmatrix} \gamma_2 \\ (n-j)\gamma_{n-j+2}\|\bar{Z}\| \end{pmatrix} + e^{\|\bar{Z}\|} \|\mathbf{v}\| C \begin{pmatrix} 1+\gamma_2 \\ \|\bar{Z}\| + (n-j)\gamma_{n-j+2}\|\bar{Z}\| \end{pmatrix} \\ & + \gamma_2 \|\mathbf{v}\| e^{\|\bar{Z}\|} (B+C) \begin{pmatrix} (1+\gamma_2) \\ \|\bar{Z}\| + (n-j)\gamma_{n-j+2}\|\bar{Z}\| \end{pmatrix}. \end{aligned} \quad (4.20)$$

4.6 Error analysis of \mathbf{v}_{new}

Let us consider

$$\text{fl}(\mathbf{v}_{\text{new}}) = ((w_1 + q_{j,1})\mathbf{a}_j * (\mathbf{1} + \bar{\mathbf{u}}_1) + (w_2 + q_{j,2})\mathbf{e}_1 * (\mathbf{1} + \bar{\mathbf{u}}_2)) * (\mathbf{1} + \bar{\mathbf{u}}_3) = \mathbf{v}_{\text{new}} + \mathbf{g}_j,$$

where

$$\mathbf{g}_j := w_1\mathbf{a}_j * \mathbf{h}_2 + q_{j,1}\mathbf{a}_j * (\mathbf{1} + \mathbf{h}_2) + w_2\mathbf{e}_1 * \mathbf{h}_2 + q_{j,2}\mathbf{e}_1 * (\mathbf{1} + \mathbf{h}_2), \quad (4.21)$$

and $|\bar{\mathbf{u}}_i| \leq u\mathbf{1}$ ($i = 1, 2, 3$), $(\mathbf{1} + \bar{\mathbf{u}}_i) * (\mathbf{1} + \bar{\mathbf{u}}_j) = \mathbf{1} + \mathbf{h}_2$. But

$$w_1 = \beta_j v_{\text{old},1} + \frac{\lambda_j}{\alpha_j^2} \mathbf{b}_j^T \mathbf{v}_{\text{old}},$$

and hence $|w_1| \leq l_1$, where

$$l_1 := e^{\|\bar{Z}\|} \|\mathbf{v}\| \left(\frac{\sinh(\sqrt{n-j}\|\bar{Z}\|)}{\sqrt{n-j}\|\bar{Z}\|} + \frac{2\sinh^2(\sqrt{n-j}\|\bar{Z}\|/2)}{\|\bar{Z}\|} \right). \quad (4.22)$$

Also, $w_2 = \lambda_j v_{\text{old},1} + \beta_j \mathbf{b}_j^T \mathbf{v}_{\text{old}}$, such that $|w_2| \leq l_2$, where

$$l_2 := e^{\|\bar{Z}\|} \|\mathbf{v}\| \left(\frac{\sinh(\sqrt{n-j}\|\bar{Z}\|)}{\sqrt{n-j}} (n-j) + 2\sinh^2(\|\bar{Z}\|/2) \right).$$

So from (4.20), (4.22), and (4.21), we have

$$|\mathbf{g}_j| \leq (l_1 \|\bar{Z}\| \gamma_2 + r_{j,1} \|\bar{Z}\| (1 + \gamma_2) + l_2 \gamma_2 + r_{j,2} (1 + \gamma_2)) * \mathbf{1}.$$

4.7 Error analysis of \mathbf{v}

$\text{fl}(\mathbf{v}(j:n)) = (\mathbf{v}_{\text{old}} * (\mathbf{1} + \mathbf{h}_2) + \mathbf{v}_{\text{new}} + \mathbf{g}_j) * (\mathbf{1} + \bar{\mathbf{u}}_4) := \mathbf{v}(j:n) + \mathbf{f}_j$, where $\mathbf{f}_j := \mathbf{v}_{\text{old}} * \mathbf{h}_3 + \mathbf{v}_{\text{new}} * \bar{\mathbf{u}}_4 + \mathbf{g}_j * (\mathbf{1} + \bar{\mathbf{u}}_4)$, and $|\bar{\mathbf{u}}_4| \leq u\mathbf{1}$, $|\mathbf{h}_3| \leq \gamma_3 \mathbf{1}$, and finally, $|\mathbf{f}_j| \leq \mathbf{error}_j$, where

$$|\mathbf{error}_j| \leq \left(\|\mathbf{v}\| e^{\|\bar{Z}\|} \gamma_3 + \frac{3}{2} \gamma_2 l_1 \|\bar{Z}\| + \frac{3}{2} \gamma_2 l_2 + r_{j,1} \|\bar{Z}\| (1 + \gamma_2) (1 + u) + r_{j,2} (1 + \gamma_2) (1 + u) \right) * \mathbf{1}. \quad (4.23)$$

5 Sensibility of Algorithm 2

As it was shown in last section and from (4.23), the computation error of $\exp(Z)\mathbf{v}$ depends on $\|\bar{Z}\|$ which is calculated by Algorithm 1. We therefore consider $\bar{Z} = Z + E$ in which the norm of E is obtained from (3.3) and (3.4). Therefore, the relations (4.3) and (4.6) reduce to $|\xi_j| \leq \varphi_j$ and $|\frac{\xi_j}{\alpha_j}| \leq \psi_j$ where

$$\begin{aligned}\varphi_j &= u\sqrt{n-j}\|Z\| + \frac{1}{\sqrt{k_0}}((n-j)\gamma_{n-j}(\|Z\| + \|E\|)^2 + (n-j)(2\|E\|\|Z\| + \|E\|^2))(1+u), \\ \psi_j &= u + \frac{1}{k_0}((n-j)\gamma_{n-j}(\|Z\| + \|E\|)^2 + (n-j)(2\|E\|\|Z\| + \|E\|^2))(1+u).\end{aligned}$$

After substituting φ_j and ψ_j in the appropriate formulas given in the last section, the error upper bounds are computed in terms of $\|E\|$. Accordingly, the estimate (4.17) becomes

$$|\varepsilon_j^4| \leq (n-j)\gamma_{n-j+2}\|Z\|\|\mathbf{v}\|e^{\|Z\|} + (n-j)(1 + \gamma_{n-j+2})\|E\|\|\mathbf{v}\|e^{\|Z\|},$$

that in turn provides the error bound given by (4.20) as follows:

$$\begin{aligned}\mathbf{r}_j &:= B \left(\begin{array}{c} \gamma_2\|\mathbf{v}\| \\ (n-j)\gamma_{n-j+2}\|Z\|\|\mathbf{v}\|e^{\|Z\|} + (n-j)(1 + \gamma_{n-j+2})\|E\|\|\mathbf{v}\|e^{\|Z\|} \end{array} \right) \\ &+ C \left(\begin{array}{c} (1 + \gamma_2)\|\mathbf{v}\| \\ (n-j)\|Z\|\|\mathbf{v}\|e^{\|Z\|} + (n-j)\gamma_{n-j+2}\|Z\|\|\mathbf{v}\|e^{\|Z\|} \\ + (n-j)(1 + \gamma_{n-j+2})\|E\|\|\mathbf{v}\|e^{\|Z\|} \end{array} \right) \\ &+ \gamma_2(B + C) \left(\begin{array}{c} (1 + \gamma_2)\|\mathbf{v}\|e^{\|Z\|} \\ (n-j)\|Z\|\|\mathbf{v}\|e^{\|Z\|} + (n-j)\gamma_{n-j+2}\|Z\|\|\mathbf{v}\|e^{\|Z\|} \\ + (n-j)(1 + \gamma_{n-j+2})\|E\|\|\mathbf{v}\|e^{\|Z\|} \end{array} \right).\end{aligned}$$

Now if we consider initial errors influencing the vectors $\mathbf{v}(j : n) = \mathbf{v}_{\text{old}} + \mathbf{v}_{\text{new}}$, for $j \geq 2$, these vectors can be represented by $\hat{\mathbf{v}}(j : n) = \mathbf{v}(j : n) + \mathbf{er}(j)$, where $\mathbf{er}(1) = 0$. Hence, the estimate (4.17) reduces to

$$|\varepsilon_j^4| \leq (n-j)\gamma_{n-j+2}\|Z\|\|\mathbf{v}\|e^{\|Z\|} + (n-j)(1 + \gamma_{n-j+2})(\|Z\|\|\mathbf{er}\| + \|E\|\|\mathbf{v}\|e^{\|Z\|} + \|E\|\|\mathbf{er}\|), \quad (5.1)$$

in which the quantities $\|E\|\|\mathbf{er}\|$, which are very small, can be neglected. Obviously, the upper bound given by (5.1) affects the variations of \mathbf{r}_j and \mathbf{error}_j . Let “error” be the final computing error committed by both Algorithms 1 and 2. In Figure 1, we compared the dependence of $\|E\|$ and $\|\mathbf{error}\|$ using 10×10 random matrices hZ with Z normalized, so that $\|Z\| = 1$, $h = \frac{1}{2^2}, \frac{1}{2^3}, \dots, \frac{1}{2^{11}}$ and 10×1 random unit vector \mathbf{v} . Figure 1 shows the stability of algorithms with respect to round off errors. Moreover, the statistical correlation coefficient between backward and forward errors is 0.9531.

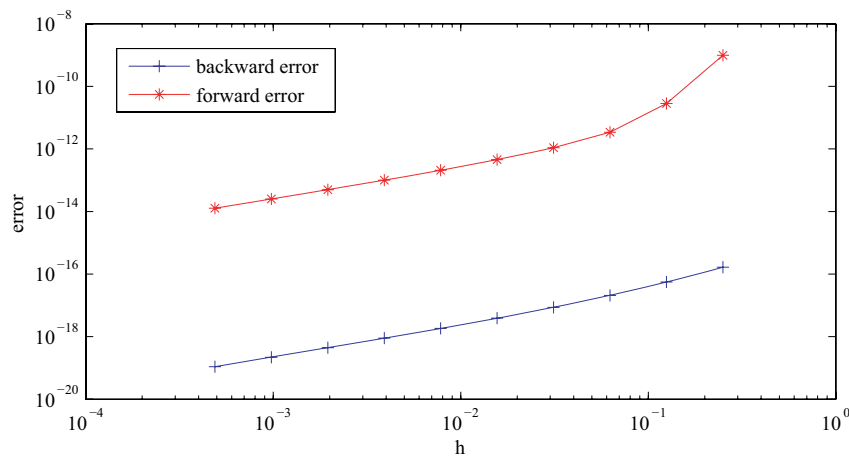


Figure 1: Backward and forward errors versus h for Algorithms 1 and 2.

6 Conclusion

As we expected, the upper bound of $|\mathbf{error}_j|$ (4.23) strongly depends on the input data Z and \mathbf{v} . More precisely, if $k_0 \gg u$, then this error bound can be controlled by $\max\{ue^{\|Z\|}, \|\mathbf{v}\|\}$. Finally, we have shown the stability of the GPD method under condition (1.1).

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