Study of Second Grade Fluid over a Rotating Disk with Coriolis and Centrifugal Forces

Shuaiba M*, Shaha RA and Khana A
Department of Basic Sciences and Islamiat, University of Engineering and Technology, Pakistan

Abstract
The steady flow of an incompressible second grade viscoelastic fluid above a rotating disk that is stretching in radial direction is studied. The Coriolis and centrifugal forces are being taken into account. The constitutive non-linear partial differential equations are transformed to system of ordinary differential equations. For analysis Homotopy Analysis Method (HAM) BVPh2.0 package is used. The effects of viscoelastic parameter α, rotating parameter β, stretching parameter γ, Coriolis and centrifugal forces on the velocity component are discussed.

Keywords: Rotating disk; Free surface flow; Rotating stretching disk; HAM mathematica package

Introduction
Rotating disk flow plays an important role in the field of engineering and industry. Centrifugal pumps are extensively used in petroleum industry to transport high viscosity fluids such as waxy crude oils. The first study of rotating disk was introduced by Von Karman in 1921 [1]. He was able to found that the rotating disk flow is a type of boundary layer flow and there is no depend of radial distance on the boundary thickness. In 1934, cochran [2], provided asymptotic solution to the ordinary differential equations derived by Von Karman. Although the analysis was simple but valuable in the field of rotating field. The work of cochran was extended by Benton [3], in 1966. He provided better solutions and solved the unsteady problem.

In recent years, much attention has been given to the rotating flow of non-Newtonian fluid concerning to its applications in industries. The steady flow of non-Newtonian fluid over rotating disk with uniform suction was considered by Mithal [4], in 1961. His solutions were valid for small values of non-Newtonian problems.

Later, Attia [5] in 2003 extended the idea of Mithal and studied the same problem to the transient state with heat transfer. Their solutions were valid for the whole range of the parameters. In addition the reader may consult [6,7] for the studies of non-Newtonian fluids.

Boundary layer flow equations are developed by Reynolds number Re → ∞ in the boundary layer region combine with the use of the order of ε² Re=1, where ε → 0.

A challenging mathematical model is developed with the non-linearity in the term involving maximum order derivation. Most of analytic methods such as Adomian Decomposition Method, Differential Transform Method, Variation Iterative Method and Optimal Homotopy Asymptotic Methods fails to solve this problem. We handle this problem by HAM BVPh2.0 package [8] using 20th-order of approximations.

The following strategy is applied to the rest of the paper. In section 3 the basic governing equations for the motion are formulated in cylindrical coordinates. Section 4 is the solution by homotopy analysis method. Section 5 is the error analysis. Section 6 contains the numerical results and their discussion for different values of physical parameters. Finally, our conclusion follows in section 7.

Formulation of the Problem
Let us consider the steady incompressible flow of a Rivlin–Erickson type fluid produced by the rotation of an insulated disk of radius R with angular speed Ω and radial stretching. The disk is stretching in radial direction which has a velocity u. The co-ordinate system (r,θ,z) is adopted whose origin is taken at the center of the disk. In which r-axis is along the radius of the disk, z-axis is perpendicular to the disk and θ is oriented in the direction of rotation. Assuming flow is laminar, axial symmetric and its density ρ, is constant.

Due to no penetration the value of w vanishes near the surface of the insulated disk. The tangential velocity v have a value Ωr at the disk surface. The position vector is given by

\[ \vec{r} = (r \cos \theta, r \sin \theta, z) \]

The basic equations governing the flow of second grade fluid is the continuity equation

\[ \nabla \cdot u = 0 \]  

(2)

and Navier-Stokes equations (NSE)

\[ \rho \left( \frac{Du}{Dt} + 2w \times \vec{v} + \vec{v} \times (\vec{u} \times \vec{r}) \right) = -\nabla \cdot \vec{p} + \nabla \cdot \vec{\tau} \]  

(3)
The continuity equation for incompressible fluid is given as

\[ \rho \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) = -\nabla p + \tau + \mathbf{F} \]

where \( \mathbf{u} \) is the velocity vector, \( \rho \) is the density, \( p \) is the pressure, \( \tau \) is the stress tensor, and \( \mathbf{F} \) is the body force vector.

The continuity equation in cylindrical coordinates has the form

\[ \frac{1}{r} \frac{\partial}{\partial r} (r \rho u_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho u_\theta) + \frac{\partial}{\partial z} (\rho u_z) = 0 \]

The centripetal force term \( 3 \) is

\[ \mathbf{F}_c = \mathbf{u} \times \mathbf{F}_C = \mathbf{u} \times \left( -r \mathbf{\hat{r}} - \omega \times \mathbf{u} \right) \]

and \( 5 \) is stress tensor in a second grade fluid given as

\[ \tau = -\frac{2}{3} \sigma \epsilon + \frac{2}{3} \tau \]

where \( \sigma \) is the Cauchy stress tensor and \( \epsilon \) is the strain rate tensor.

The momentum equation in cylindrical coordinates for the \( r \)-component is

\[ \rho \frac{\partial}{\partial t} (r u_r) + \frac{\partial}{\partial r} (r \rho u_r u_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho u_r u_\theta) + \frac{\partial}{\partial z} (\rho u_r u_z) = -\frac{\partial p}{\partial r} + \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (\tau_{r\theta} r) + \frac{\partial}{\partial z} (\tau_{rz}) + F_r \]

and for the \( \theta \)-component is

\[ \rho \frac{\partial}{\partial t} (r u_\theta) + \frac{\partial}{\partial r} (r \rho u_\theta u_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho u_\theta u_\theta) + \frac{\partial}{\partial z} (\rho u_\theta u_z) = \frac{\partial p}{\partial \theta} - \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} (\tau_{r\theta} r) + \frac{\partial}{\partial z} (\tau_{r\theta}) + F_\theta \]

The continuity equation for non-inertial cylindrical coordinates has the form

\[ \frac{1}{r} \frac{\partial}{\partial r} (r \rho u_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho u_\theta) + \frac{\partial}{\partial z} (\rho u_z) + \frac{1}{r} \frac{\partial p}{\partial \theta} = 0 \]

where \( \rho \) is the density, \( u_r, u_\theta, u_z \) are the radial, azimuthal, and axial velocity components, respectively, and \( p \) is the pressure.

The centripetal term \( 3 \) is

\[ \mathbf{F}_c = -r \mathbf{\hat{r}} - \omega \times \mathbf{u} \]

and \( 5 \) is stress tensor in a second grade fluid given as

\[ \tau = -\frac{2}{3} \sigma \epsilon + \frac{2}{3} \tau \]

where \( \sigma \) is the Cauchy stress tensor and \( \epsilon \) is the strain rate tensor.

The centripetal term \( 3 \) is

\[ \mathbf{F}_c = -r \mathbf{\hat{r}} - \omega \times \mathbf{u} \]

The continuity equation in cylindrical coordinates for the \( r \)-component is

\[ \frac{1}{r} \frac{\partial}{\partial r} (r \rho u_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho u_\theta) + \frac{\partial}{\partial z} (\rho u_z) + \frac{1}{r} \frac{\partial p}{\partial \theta} = 0 \]

where \( \rho \) is the density, \( u_r, u_\theta, u_z \) are the radial, azimuthal, and axial velocity components, respectively, and \( p \) is the pressure.

The centripetal term \( 3 \) is

\[ \mathbf{F}_c = -r \mathbf{\hat{r}} - \omega \times \mathbf{u} \]

and \( 5 \) is stress tensor in a second grade fluid given as

\[ \tau = -\frac{2}{3} \sigma \epsilon + \frac{2}{3} \tau \]

where \( \sigma \) is the Cauchy stress tensor and \( \epsilon \) is the strain rate tensor.

The centripetal term \( 3 \) is

\[ \mathbf{F}_c = -r \mathbf{\hat{r}} - \omega \times \mathbf{u} \]

The continuity equation in cylindrical coordinates for the \( r \)-component is

\[ \frac{1}{r} \frac{\partial}{\partial r} (r \rho u_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho u_\theta) + \frac{\partial}{\partial z} (\rho u_z) + \frac{1}{r} \frac{\partial p}{\partial \theta} = 0 \]

where \( \rho \) is the density, \( u_r, u_\theta, u_z \) are the radial, azimuthal, and axial velocity components, respectively, and \( p \) is the pressure.

The centripetal term \( 3 \) is

\[ \mathbf{F}_c = -r \mathbf{\hat{r}} - \omega \times \mathbf{u} \]

and \( 5 \) is stress tensor in a second grade fluid given as

\[ \tau = -\frac{2}{3} \sigma \epsilon + \frac{2}{3} \tau \]

where \( \sigma \) is the Cauchy stress tensor and \( \epsilon \) is the strain rate tensor.

The centripetal term \( 3 \) is

\[ \mathbf{F}_c = -r \mathbf{\hat{r}} - \omega \times \mathbf{u} \]
The continuity equation satisfied identically and momentum equations take the following form after algebraic manipulation

\[ f^2 - 2f'g' + 2f'' - 4f[1 + 2f'[1 + 0 + 4f + 4g^2 - f' + 2f''^2] = 0 \]  

\[ f'g' \geq g' + \beta f' + \alpha \left[-g'f' + 2g'f' - gf'' + fg'' \right] = 0 \]  

\[ f'' + 4g'g'' = 0 \]  

Boundary conditions becomes

\[ f'(0) = -2g', \quad f(0) = 0, \quad g(0) = 1, \quad f'(\infty) = 0, \quad g(\infty) = 0 \]  

where \( \alpha = \frac{\Omega}{\delta} \) is viscoelastic parameter, \( \beta = \Omega \delta' v \) is the rotational number, \( E_s = \frac{\mu}{\Omega \delta} \) is Ekman number and \( S \) is the height.

The model applies strictly to an infinite disk, but can be applied to a finite disk of radius \( R \), provided that \( R = \delta \) is satisfied.

**Solution by Homotopy Analysis Method**

By the HAM method, the functions \( f(z) \) and \( g(z) \) as:

\[ f_m(z) = \sum_{n=0}^{\infty} \alpha_{m,n} z^n \exp(-nz), \]

\[ g_m(z) = \sum_{n=0}^{\infty} \beta_{m,n} z^n \exp(-nz) \]

where \( \alpha_{m,n} \) and \( \beta_{m,n} \) are the coefficients to be determined. Initial guess and auxiliary linear operator are chosen as follows:

\[ f_0(z) = r(z) \]  

\[ g_0(z) = \bar{z} \]  

The above auxiliary linear operators have the following properties

\[ L \left( c_i + c_i e^z \right) = 0, \]

\[ L \left( c_i + c_i e^{-z} \right) = 0, \]

where \( c_i (i = 1 - 5) \) are arbitrary constants. The zeroth order deformation problems can be obtain as:

\[ (1-q) L_i \left[ f(z) - f_0(z) \right] = q h_i N_i \left[ f(z) ; q \right], \]

\[ (1-q) L_i \left[ g(z) - g_0(z) \right] = q h_i N_i \left[ g(z) ; q \right], \]

where \( q, h_i, N_i \) are the non-zero auxiliary parameter and auxiliary linear parameter and \( N_i \) is nonlinear operators.

To convert the partial differential equation into ordinary differential equation we make the following transformations.

\[ u = \frac{r}{2} f'(z), \quad v = r g(z), \quad w = f(z) \]  

The continuity equation satisfied identically and momentum equations take the following form after algebraic manipulation

\[ f^2 - 2f'g' + 2f'' - 4f[1 + 2f'[1 + 0 + 4f + 4g^2 - f' + 2f''^2] = 0 \]  

\[ f'g' \geq g' + \beta f' + \alpha \left[-g'f' + 2g'f' - gf'' + fg'' \right] = 0 \]  

\[ f'' + 4g'g'' = 0 \]  

Boundary conditions becomes

\[ f'(0) = -2g', \quad f(0) = 0, \quad g(0) = 1, \quad f'(\infty) = 0, \quad g(\infty) = 0 \]  

where \( \alpha = \frac{\Omega}{\delta} \) is viscoelastic parameter, \( \beta = \Omega \delta' v \) is the rotational number, \( E_s = \frac{\mu}{\Omega \delta} \) is Ekman number and \( S \) is the height.

The model applies strictly to an infinite disk, but can be applied to a finite disk of radius \( R \), provided that \( R = \delta \) is satisfied.
For q=0 and q=1 we have:
\[ \tilde{f}(z;0) = f_0(z), \quad \tilde{f}(z;1) = f(z) \]
\[ \tilde{g}(z;0) = g_0(z), \quad \tilde{g}(z;1) = g(z). \]

Therefore, as the embedding parameter \( q \) increases from 0 to 1, \( \tilde{f}(z;\xi) \) and \( \tilde{g}(z;\xi) \) vary from their initial guesses \( f_0(z) \) and \( g_0(z) \) to the exact solutions \( f(z) \) and \( g(z) \), respectively. Taylor’s series expansion of these functions yields:
\[ f(z;q) = f_0(z) + \sum_{n=1}^{\infty} f_n(z) q^n, \]
\[ g(z;q) = g_0(z) + \sum_{n=1}^{\infty} g_n(z) q^n, \]
where
\[ f_n = 1 \frac{\partial^n f(z;q)}{\partial q^n} \bigg|_{q=0}, \quad g_n = 1 \frac{\partial^n g(z;q)}{\partial q^n} \bigg|_{q=1}. \]

Keeping in mind the above series depends on \( h_1 \) and \( h_2 \). On the assumption that the non-zero auxiliary parameters are chosen so that Eq.(39) converge at \( q=1 \).

Therefore we can obtain:
\[ f(z) = f_0(z) + \sum_{n=1}^{\infty} f_n(z), \]
\[ g(z) = g_0(z) + \sum_{n=1}^{\infty} g_n(z). \]

Differentiating m-times the zeroth order deformation equations (35) and (36) one has the mth order deformation equations as:
\[ L_f[f_m(z) - X_m f_{m-1}(z)] = h_1 R_{f_m}(z), \]
\[ L_g[g_m(z) - X_m g_{m-1}(z)] = h_2 R_{g_m}(z), \]
where, the boundary conditions (29) takes the form
\[ f_n(0) = f_n(0) = f_n(\infty) = 0, \]
\[ g_n(0) = g_n(\infty), \]
\[ R_{f_m}(z) = \sum_{j=0}^{\infty} f_j(z) \frac{\partial^{m-j} f_{m-j}(z) \partial z^j}{\partial z^j}, \]
\[ R_{g_m}(z) = \sum_{j=0}^{\infty} g_j(z) \frac{\partial^{m-j} g_{m-j}(z) \partial z^j}{\partial z^j}, \]
\[ \alpha_1 = \sum_{j=0}^{\infty} f_j(z) \frac{\partial^{m-j} f_{m-j}(z) \partial z^j}{\partial z^j}, \]
\[ \alpha_2 = \sum_{j=0}^{\infty} g_j(z) \frac{\partial^{m-j} g_{m-j}(z) \partial z^j}{\partial z^j}, \]
\[ \alpha_3 = \sum_{j=0}^{\infty} f_j(z) \frac{\partial^{m-j} f_{m-j}(z) \partial z^j}{\partial z^j}. \]

Finally, the general solution may be written as follows:
\[ f_m(z) = f'_m + c_1 e^{-z} + c_2 e^{-2z}, \]
\[ g_m(z) = g'_m + c_1 e^{-z}, \]
Where \( f'_m \) and \( g'_m \) are the special solutions.

**Error Analysis**

To perform analysis of the problem under consideration we made the first error analysis to make sure that our analysis are reliable up to the scale of minimum residual error. Before to discuss and give physical predictions we perform error analysis to investigate the validity of the HAM techniques. For this purpose, Figures 2 and Tables 1 and 2 are made. Table 1 represents the nonzero auxiliary convergence control parameters \( h_1 \) and \( h_2 \) and the minimum values of total averaged squared residual errors executed for different orders of approximations. The total squared residual error \( \xi_{m}^2 \) can be minimized by increasing the order of approximations. Here, it can be seen that increasing the order of approximations the total squared residual errors are reduced. Table 2 illustrate the individual average squared residual error at different orders of approximations. Besides this Figure 2 also shows the maximum average squared residual error at different orders of approximation. It
can also be observed that the total averaged squared errors and average squared residual errors are decreasing as the order of approximation is increasing for different values of physical parameters $\alpha$ and $\beta$.

**Results and Discussion**

In this section, we present graphical results of the system of coupled nonlinear ODE’s given in eqn. (26) and eqn. (27) corresponding the boundary conditions (28). Numerical Solution is obtained by means of the BVPh2.0, a HAM Mathematica package [9,10]. For better analysis, Figures 2-16 are plotted. In order to get the numerical solutions of the above equations, it was translated into BVPh 2.0 program in Mathematica by setting the required error $10^{-10}$. The semi infinite domain $z \in [0,\infty)$ is replaced by a finite domain $z \in [0,z_{\infty}]$. In practice, $z_{\infty}$ should be chosen sufficiently large so that the numerical solution closely approximates the terminal boundary conditions.

Figures 3-5 show the influence of the viscoelastic parameter $\alpha$ on the non dimensional axial velocity component $f(z)$ for fixed values of $\gamma=0.01$ and $\beta=0.1$, $0.5$, $5$, respectively. It is observed that increasing $\alpha$ the speed of the flow reduces. This is due to the fact, that increasing non-Newtonian effect $\alpha$ shear forces increases in the fluid domain which reduced the speed of flow. Also it can be noticed that large values of rotation number $\beta$ dominate the influence of viscoelastic parameter $\alpha$, while the effect of $\alpha$ can be seen only for small values of $\beta$ [11-15].
The variation of dimensionless axial velocity component $f(z)$ versus axial direction $z$ for different values of rotation number $\beta$ are plotted in Figs. 6-8, for fixed values of slip parameter $\gamma=0.1, 0.5$ and $\gamma=0.01$, respectively. It can be seen that increasing the value of $\beta$ results in an increase in the axial velocity component near the disk; however, as expected, the rate of increase of the axial velocity is negligible far away from the disk.

To investigate the fluid velocity along azimuthal direction with and without Coriolis and centrifugal force Figure 9 is made. Near the disk there is no effect on the velocity component $f$ and far away these forces effect $f(z)$, the combine effect of these forces clearly effect the velocity field $f$. As expected, near the disk the Coriolis and centrifugal forces balance the effect of each other [16-25].

Figures 10-12 depict the effect of viscoelastic parameter $\alpha$ rotation parameter $\beta$ and slip parameter $\gamma$ for selected values of rotation number $\beta$ and slip parameter $\gamma$. One can see that increasing non-Newtonian effect the boundary layer region decreases and increasing rotation number the boundary layer increases and similar case is seen for increasing slip parameter $\gamma$. The influence of centrifugal and Coriolis forces on radial velocity component are graphed in Figure 13.
It is interesting to note that the radial velocity increases by taking the effect of these forces. The effect of centrifugal force near the surface of the disk is seems to be negligible and far away from the surface of the rotating disk its effect can be seen clearly. The influence of different values of viscoelastic parameter \( \alpha = 0.1, 0.5, 0.8 \) for fixed values of \( \beta = 0.5 \) and \( \gamma = 0.01 \) on radial velocity component are plotted in Figure 14. It is also interesting to note that increasing the parameter \( \alpha \) the azimuthal velocity component also increases but for small values of \( \beta \) and \( \gamma \) this increase is small in magnitude [26-32].

Table 3 and 4 are made to observed the variation of azimuthal velocity component for selected values of \( \alpha \) and \( \gamma \) for the fixed values of other parameters of interest. Here it can be seen that on increasing \( \alpha \) and \( \gamma \) the azimuthal component of velocity \( g(z) \) also increases. Three dimensional radial and azimuthal velocity components are plotted in Figures 15 and 16 for different values of \( \alpha = 0.1, \beta = 0.5 \) and \( \gamma = 0.01 \).

**Conclusion**

Three dimensional rotating flow over a long disk of a viscoelastic fluid under the influence of Coriolis and centrifugal forces are studied. From the context of transformations and dimensional analysis mathematical system of ODE’s are obtained. A careful analysis of the flow is carried out by means of HAM Mathematica package. The following conclusions are made during analysis:
1. It is concluded that increasing non-Newtonian parameter $\alpha$, the radial component and axial component of velocity increases, while the azimuthal component of velocity decreases.

2. It is also concluded that increasing the rotation number $\beta$, the radial and axial components of velocity profile also increases.

3. Further more, increasing slip effect causes the radial velocity component to increase.

4. Moreover, the radial and axial velocity component increase by taking the effect of centrifugal and Coriolis forces in the momentum equations. These finding abrute and enrich our understanding about the boundary layer flow of second grade fluid.

References


