

Symmetries of the Canonical Geodesic Equations of Five-Dimensional Nilpotent Lie Algebras

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Abstract

Symmetries of the canonical geodesic equations of indecomposable nilpotent Lie groups of dimension five are constructed. For each case, the associated system of geodesics is provided. In addition, a basis for the associated Lie algebra of symmetries as well as the corresponding non-zero Lie brackets are listed and classified.

Keywords: Lie symmetry • Lie group • Canonical connection • System of geodesic equation

Introduction

Any Lie group comes equipped with a natural linear torsion-free connection. This connection was originally introduced in 1926 by Cartan and Schouten [1]. A summary of the salient features of this connection may be found in [2]. Recently, two of the current authors have studied the Lie symmetry algebras of the canonical geodesic equations for Lie groups in low dimensions. In [2] and [3], we investigated the Lie symmetries of the canonical geodesic system of indecomposable Lie groups in dimensions two, three, and four. Such a system of ODE's has also recently been encountered in the context of the inverse problem of Lagrangian mechanics [4-6].

The aim of the present paper is to investigate the Lie symmetry properties of the geodesic systems of five-dimensional indecomposable nilpotent Lie groups whose associated Lie algebras are listed in [7]. It turns out that there are six such Lie algebras that are mutually not isomorphic and furthermore, unlike many of the low-dimensional solvable Lie algebras, are "atomic", in the sense that they do belong to continuous families that depend on parameters.

The corresponding geodesic systems of equations for each of the nilpotent Lie groups were constructed in [6]. Investigation of Lie symmetries in each of these six cases is an extremely labor-intensive process, the details of which would run to many pages if presented in full and in any case the calculations are performed using MAPLE. As a consequence, we completely forgo the basic computations and are content to render compactly the conclusions. In each of these cases, we methodically provide the non-zero brackets of the original Lie algebra, the associated system of geodesics, a basis for the associated Lie algebra of symmetries and the corresponding non-vanishing Lie brackets.

It turns out that of the six nilpotent Lie algebras that are considered, two have flat canonical connections. Indeed, in both cases we provide a change of coordinates so that the geodesic equations describe the motion of a "free particle". However, it is to be emphasized that such a change of coordinates

is not compatible with the Lie algebra structure. Nonetheless, it follows that in these two cases the Lie symmetry algebra must be $\mathfrak{sl}(7, \mathbb{R})$.

One final qualitative remark is in order. We observe that, roughly speaking, the dimension of the Lie symmetry algebras of nilpotent Lie groups seems to be larger than that of comparable solvable algebras. Indeed, we have already observed that two cases lead to flat connections and so provide symmetry algebras of maximal dimension. We believe that there may be two underlying reasons that help to explain this phenomenon. First of all, the nilpotent algebras, at least in dimension five do not depend on parameters. Secondly, it appears as though the geodesic systems for nilpotent Lie algebras, always contain several trivial geodesic equations, that is, where the right hand side is zero. We believe that this circumstance deserves to be further investigated.

Free Particle Systems

In this Section, we shall review the Lie symmetries of a free particle system, being the most extreme case of a flat connection. However, this case of course transcends issues of any Lie group structure. The same results have been rediscovered many times, but we shall refer to [8] as one source. The geodesic equations will be written as

$$\ddot{x}^i = 0, \quad (1)$$

where (x^i) are a system of local coordinates on some manifold M . It will be helpful to define the dilation vector field Δ on M by

$$\Delta = tD_t + x^i D_i, \quad (2)$$

where D_i denotes the partial derivative operator with respect to x^i and there is a sum over i from 1 to n , the latter being the dimension of M . Then the following vector fields comprise a basis for the space of Lie symmetries of eqn. (1):

$$D_t, D_p, tD_p, x^i D_p, tD_p x^i D_p, t\Delta, x^i \Delta. \quad (3)$$

Adding up, we obtain a space of dimension $n^2 + 4n + 3 = (n+2)^2 - 1$ and indeed we obtain a representation of the simple Lie algebra $\mathfrak{sl}(n+2, \mathbb{R})$.

5-Dimensional Nilpotent Lie Algebras, Geodesics, and Symmetry Algebras

$$A_{5,1} : [e_3, e_5] = e_1, [e_4, e_5] = e_2.$$

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System of geodesic equations:

$$\ddot{q} = \dot{y}\dot{w},$$

$$\ddot{x} = \dot{z}\dot{w},$$

$$\ddot{y} = 0,$$

$$\ddot{z} = 0,$$

$$\ddot{w} = 0.$$

We make a change of variable to equations $\ddot{q}$ and $\ddot{x}$ so the system can be written as the free particle system. Thus,

$$\bar{q} = q - \frac{1}{2}yw \Rightarrow \ddot{\bar{q}} = 0,$$

and

$$\bar{x} = x - \frac{1}{2}zw \Rightarrow \ddot{\bar{x}} = 0.$$

Hence, the geodesic equations become

$$\ddot{\bar{q}} = 0,$$

$$\ddot{\bar{x}} = 0,$$

$$\ddot{y} = 0,$$

$$\ddot{z} = 0,$$

$$\ddot{w} = 0.$$

The symmetry Lie algebra is $\mathfrak{sl}(7, \mathbb{R})$ as was explained in Section 2.

$$A_{5,2} : [e_2, e_5] = e_1, [e_3, e_5] = e_2, [e_4, e_5] = e_3,$$

System of geodesic equations:

$$\ddot{q} = \dot{x}\dot{w},$$

$$\ddot{x} = \dot{y}\dot{w},$$

$$\ddot{y} = \dot{z}\dot{w},$$

$$\ddot{z} = 0,$$

$$\ddot{w} = 0.$$

Symmetry algebra basis and non-vanishing brackets are, respectively:

$$e_1 = D_x, e_2 = tD_q, e_3 = D_q, e_4 = D_y, e_5 = D_x, e_6 = D_z, e_7 = D_w + \frac{x D_q + y D_x + z D_y}{2},$$

$$e_8 = zD_x, e_9 = zD_q, e_{10} = wD_q, e_{11} = wD_x, e_{12} = yD_{q+xD_x}, e_{13} = twD_q + 2tD_x,$$

$$e_{14} = \frac{1}{2}w^2D_q + wD_x, e_{15} = wzD_q + 2zD_x, e_{16} = (zw - 2y)D_x,$$

$$e_{17} = \frac{1}{6}w^3D_q + \frac{1}{2}w^2D_x + wD_y, e_{18} = \frac{1}{24}w^4D_q + \frac{1}{6}w^3D_x + \frac{1}{2}w^2D_y + wD_z,$$

$$e_{19} = (yw - \frac{1}{2}zw^2)D_q + (2y - zw)D_x, e_{20} = tD_x, e_{21} = qD_q + xD_x + yD_y + zD_z,$$

$$e_{22} = \frac{w}{24}(w^3D_q + 4w^2D_x + 12wD_y + 24D_z), e_{23} = \frac{(2y - wz)}{2}(2D_x + wD_q),$$

$$e_{24} = \frac{(w^3z - 3w^2y + 6wx - 12q)}{24}D_q + \frac{(wz - 2y)}{2}(2D_y + wD_x),$$

$$e_{25} = (\frac{zw^4}{24} + \frac{xw^2}{2} - \frac{yw^2}{6} - qw)D_q + (\frac{zw^3}{6} + wx - \frac{yw^2}{2} - 2q)D_x + (\frac{zw^2}{2} - yw)D_y + (zw - 2y)D_z.$$

$$[e_1, e_2] = e_3, [e_1, e_{13}] = e_{10} + 2e_5, [e_1, e_{20}] = e_{17}, [e_2, e_8] = -e_9, [e_2, e_{22}] = -e_9, [e_2, e_{24}] = -e_9, [e_2, e_{25}] = -e_{10},$$

$$[e_2, e_{16}] = 2e_{12} - e_{15}, [e_2, e_{20}] = -e_{21}, [e_2, e_{21}] = e_{21}, [e_2, e_{22}] = 2e_{21}, [e_2, e_{24}] = -e_{21},$$

$$[e_2, e_{25}] = -e_{13}, [e_3, e_{21}] = e_3, [e_3, e_{22}] = 2e_3, [e_3, e_{24}] = -e_3, [e_3, e_{25}] = 2e_5,$$

$$[e_4, e_7] = \frac{e_3}{2}, [e_4, e_{12}] = e_3, [e_4, e_{16}] = -2e_{11}, [e_4, e_{19}] = e_{10} + 2e_5, [e_4, e_{21}] = e_4,$$

$$[e_4, e_{22}] = -\frac{e_{14}}{2}, [e_4, e_{23}] = -\frac{e_{10}}{2}, [e_4, e_{24}] = -e_{14} - e_4, [e_4, e_{25}] = -e_{17} - 2e_6, [e_5, e_7] = \frac{e_3}{2},$$

$$[e_5, e_{21}] = e_5, [e_5, e_{22}] = \frac{e_{10}}{2} + 2e_3, [e_5, e_{23}] = e_3, [e_5, e_{24}] = e_{10} + e_5, [e_5, e_{25}] = e_{14},$$

$$[e_6, e_7] = \frac{e_4}{2}, [e_6, e_8] = e_1, [e_6, e_9] = e_3, [e_6, e_{12}] = e_5, [e_6, e_{15}] = e_{10} + 2e_5, [e_6, e_{16}] = e_{11},$$

$$[e_6, e_{19}] = -e_{14}, [e_6, e_{21}] = e_6, [e_6, e_{22}] = \frac{e_{17}}{2}, [e_6, e_{23}] = e_4 + \frac{e_{14}}{2}, [e_6, e_{24}] = e_{17} + e_6,$$

$$[e_6, e_{25}] = e_{18}, [e_7, e_{10}] = e_3, [e_7, e_{11}] = e_1,$$

$$[e_7, e_{14}] = e_5 + \frac{e_{10}}{2}, [e_7, e_{17}] = e_4 + \frac{e_{14}}{2}, [e_7, e_{18}] = e_6 + \frac{e_{17}}{2},$$

$$[e_7, e_{22}] = e_7, [e_8, e_{13}] = e_{15}, [e_8, e_{18}] = -e_{11}, [e_8, e_{20}] = e_8, [e_8, e_{21}] = -e_8, [e_8, e_{24}] = -e_8,$$

$$[e_8, e_{25}] = -e_{16},$$

$$[e_9, e_{18}] = -e_{10}, [e_9, e_{22}] = 2e_9, [e_9, e_{24}] = -2e_9, [e_9, e_{25}] = 2e_{12} - 2e_{15}, [e_{10}, e_{21}] = e_{10},$$

$$[e_{10}, e_{22}] = e_{10},$$

$$[e_{10}, e_{24}] = -e_{10}, [e_{10}, e_{25}] = -2e_{14}, [e_{11}, e_{13}] = 2e_{14}, [e_{11}, e_{20}] = e_{11}, [e_{11}, e_{22}] = -e_{11},$$

$$[e_{12}, e_{17}] = -e_{10}$$

$$[e_{12}, e_{18}] = -e_{14}, [e_{12}, e_{22}] = 2e_{12}, [e_{13}, e_{16}] = 2e_{19}, [e_{13}, e_{20}] = -e_{13}, [e_{13}, e_{21}] = e_{13},$$

$$[e_{13}, e_{22}] = 2e_{13},$$

$$[e_{13}, e_{23}] = 2e_{21}, [e_{13}, e_{24}] = e_{13}, [e_{14}, e_{21}] = e_{14}, [e_{14}, e_{22}] = e_{14}, [e_{14}, e_{23}] = e_{10},$$

$$[e_{14}, e_{24}] = e_{14}, [e_{15}, e_{18}] = -2e_{14},$$

$$[e_{15}, e_{22}] = 2e_{15}, [e_{15}, e_{23}] = 2e_9, [e_{15}, e_{25}] = 2e_{19}, [e_{16}, e_{17}] = 2e_{11}, [e_{16}, e_{20}] = e_{16},$$

$$[e_{16}, e_{21}] = -e_{16},$$

$$[e_{16}, e_{23}] = 2e_8, [e_{16}, e_{24}] = e_{16}, [e_{17}, e_{19}] = 2e_{14}, [e_{17}, e_{21}] = e_{17}, [e_{17}, e_{22}] = -$$

$$e_{17}, [e_{17}, e_{24}] = -e_{17}, [e_{17}, e_{25}] = -2e_{18},$$

$$[e_{18}, e_{21}] = e_{18}, [e_{18}, e_{22}] = -e_{18}, [e_{18}, e_{23}] = e_{17}, [e_{18}, e_{24}] = e_{18}, [e_{19}, e_{22}] = 2e_{19}, [e_{19}, e_{23}] =$$

$$2e_{12} - 2e_{15}, [e_{19}, e_{24}] = 2e_{19}, [e_{23}, e_{24}] = -2e_{23}, [e_{19}, e_{25}] = -2e_{24}, [e_{24}, e_{25}] = -2e_{25}.$$

We observe that the symmetry algebra is a 25-dimensional indecomposable algebra which has a non-trivial Levi decomposition $\mathfrak{sl}(2, \mathbb{R}) \ltimes (\mathbb{R}^3 \ltimes \mathbb{R}^{19})$. The radical comprises a non-abelian nilradical spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}, e_{17}, e_{18}, e_{19}$ and the complement to the nilradical is abelian and spanned by e_{20}, e_{21}, e_{22} . The $\mathfrak{sl}(2, \mathbb{R})$ part is spanned by e_{23}, e_{24}, e_{25} .

$$A_{5,3} : [e_3, e_4] = e_2, [e_3, e_5] = e_1, [e_4, e_5] = e_3.$$

System of geodesic equations:

$$\ddot{q} = -2z\dot{z}\dot{w},$$

$$\ddot{x} = 2\dot{y}\dot{w},$$

$$\ddot{y} = \dot{z}\dot{w},$$

$$\ddot{z} = 0,$$

$$\ddot{w} = 0.$$

Symmetry algebra basis and non-vanishing brackets are, respectively:

$$e_1 = D_x, e_2 = tD_q, e_3 = D_q, e_4 = tD_x, e_5 = D_x, e_6 = D_z, e_7 = D_w, e_8 = zD_x, e_9 = wD_x, e_{10} = zD_q, e_{11} = zD_x,$$

$$e_{12} = wD_x, e_{13} = wD_x, e_{14} = yD_x + \frac{zD_y}{2}, e_{15} = zwD_x + zD_y, e_{16} = -zwD_q + D_z,$$

$$e_{17} = \frac{w^2D_x}{2} + \frac{wD_y}{2}, e_{18} = (y - \frac{zw}{2})D_q, e_{19} = \frac{w^2D_x}{2} + \frac{wD_y}{2}, e_{20} = (y - \frac{zw}{2})D_q,$$

$$e_{25} = (\frac{zw^4}{24} + \frac{xw^2}{2} - \frac{yw^2}{6} - qw)D_q + (\frac{zw^3}{6} + wx - \frac{yw^2}{2} - 2q)D_x + (\frac{zw^2}{2} - yw)D_y + (zw - 2y)D_z.$$

$$\begin{aligned}
 e_{19} &= tD_t + (yw - \frac{w^2z}{2})D_x - (y - \frac{zw}{2})D_y, e_{20} = qD_q + 2xD_x + yD_y + wD_w, \\
 e_{21} &= tD_t - (yw - \frac{w^2z}{2})D_x - (y - \frac{zw}{2})D_y, e_{22} = tD_x + tD_y, e_{23} = (zw - 2y)D_t, \\
 e_{24} &= qD_q - wD_w - xD_x + zD_z, e_{25} = zD_w - \frac{z^3D_q}{3} + \frac{z^2D_y}{2} + (yz - \frac{q}{2})D_x, \\
 e_{26} &= (2yw - 2x - w^2z)D_q + \frac{w^3D_x}{3} + \frac{w^2D_y}{2} + wD_z. \\
 [e_1, e_2] &= e_3, [e_1, e_4] = e_5, [e_1, e_{19}] = e_1, [e_1, e_{21}] = e_1, [e_1, e_{22}] = e_{12} + e_6, [e_2, e_8] = -e_{10}, \\
 [e_2, e_{13}] &= -e_9, [e_2, e_{19}] = -e_2, [e_2, e_{20}] = 3e_2, [e_2, e_{21}] = -e_2, [e_2, e_{23}] = 2e_{18}, [e_2, e_{24}] = e_2, \\
 [e_2, e_{25}] &= -\frac{e_4}{2}, [e_3, e_{20}] = 3e_3, [e_3, e_{24}] = e_3, [e_3, e_{25}] = -\frac{e_5}{2}, [e_4, e_8] = - \\
 e_{11}, [e_4, e_{13}] &= -e_{12}, \\
 [e_4, e_{19}] &= -e_4, [e_4, e_{20}] = 3e_4, [e_4, e_{21}] = -e_4, [e_4, e_{23}] = 2e_{14} - e_{15}, [e_4, e_{24}] = - \\
 e_4, [e_4, e_{26}] &= -2e_2, \\
 [e_5, e_{20}] &= 3e_5, [e_5, e_{24}] = -e_5, [e_4, e_{26}] = -2e_3, [e_6, e_{14}] = e_5, [e_6, e_{18}] = e_3, \\
 [e_6, e_{19}] &= e_{12} + e_6, \\
 [e_6, e_{20}] &= -2e_{12}, [e_6, e_{21}] = -e_{12} - e_6, [e_6, e_{23}] = -2e_1, [e_6, e_{25}] = e_{11}, [e_6, e_{26}] = 2e_9, \\
 [e_7, e_9] &= e_3, \\
 [e_7, e_{12}] &= e_5, [e_7, e_{13}] = e_1, [e_7, e_{15}] = e_{11}, [e_7, e_{16}] = -e_{10}, \\
 [e_7, e_{17}] &= e_{12} + \frac{e_6}{2}, [e_7, e_{18}] = -\frac{e_{10}}{2}, \\
 [e_7, e_{19}] &= e_{14} - e_{15}, [e_7, e_{20}] = 2e_{15} - 2e_{14} + e_7, [e_7, e_{21}] = -e_{14} + e_{15}, [e_7, e_{22}] = e_4, \\
 [e_7, e_{23}] &= e_8, \\
 [e_7, e_{24}] &= -e_7, [e_7, e_{26}] = 2e_{18} + e_{16} + 2e_{17}, [e_8, e_{16}] = -e_1, [e_8, e_{19}] = e_8, [e_8, e_{20}] = -e_8, \\
 [e_7, e_{21}] &= e_8, \\
 [e_8, e_{22}] &= e_{15}, [e_8, e_{24}] = -e_8, [e_8, e_{26}] = -e_{13}, [e_9, e_{20}] = 2e_9, [e_9, e_{24}] = -2e_9, \\
 [e_9, e_{25}] &= -e_{10} - \frac{e_{12}}{2}, \\
 [e_{10}, e_{16}] &= -e_3, [e_{10}, e_{20}] = 2e_{10}, [e_{10}, e_{25}] = -\frac{e_{11}}{2}, [e_{10}, e_{26}] = -e_9, [e_{11}, e_{16}] = -e_5, \\
 [e_{11}, e_{20}] &= 2e_{11}, \\
 [e_{11}, e_{24}] &= -2e_{11}, [e_{11}, e_{26}] = -2e_{10} - e, [e_{12}, e_{20}] = 2e_{12}, [e_{12}, e_{25}] = -e_{11}, [e_{12}, e_{26}] = -2e_9, \\
 [e_{13}, e_{19}] &= e_{13}, [e_{13}, e_{20}] = -e_{13}, [e_{13}, e_{21}] = e_{13}, [e_{13}, e_{22}] = 2e_{17}, [e_{13}, e_{24}] = e_{13}, \\
 [e_{13}, e_{25}] &= -e_8, \\
 [e_{14}, e_{19}] &= -e_{14} + e_{15} = -e_{11}, [e_{14}, e_{16}] = -\frac{e_6}{2}, [e_{14}, e_{17}] = -\frac{e_{12}}{2}, [e_{14}, e_{18}] = \frac{e_{10}}{2}, \\
 [e_{14}, e_{20}] &= 3e_{14} - 2e_{15}, [e_{14}, e_{21}] = e_{14} - e_{15}, [e_{14}, e_{22}] = -e_4, [e_{14}, e_{23}] = -e_8, [e_{14}, e_{24}] = -e_{14}, \\
 [e_{14}, e_{26}] &= -e_{17} - 2e_{18}, [e_{15}, e_{16}] = -e_{12} - e_6, [e_{15}, e_{18}] = e_{10}, [e_{15}, e_{19}] = e_{15}, [e_{15}, e_{20}] = -e_{15}, \\
 [e_{15}, e_{21}] &= -e_{15}, [e_{15}, e_{23}] = -2e_8, [e_{15}, e_{24}] = -e_{15}, [e_{15}, e_{26}] = -2e_{17}, [e_{16}, e_{18}] = -\frac{e_9}{2}, \\
 [e_{16}, e_{19}] &= -e_{17}, \\
 [e_{16}, e_{20}] &= e_{16} + 2e_{17}, [e_{16}, e_{21}] = e_{17}, [e_{16}, e_{23}] = e_{13}, [e_{16}, e_{24}] = e_{16}, [e_{16}, e_{25}] = \frac{e_{15}}{2} + e_{14}, \\
 e_7, [e_{17}, e_{18}] &= \frac{e_9}{2} \\
 [e_{17}, e_{19}] &= e_{17}, [e_{17}, e_{20}] = -e_{17}, [e_{17}, e_{23}] = -e_{13}, [e_{17}, e_{24}] = e_{17}, [e_{17}, e_{25}] = -\frac{e_{15}}{2}, \\
 [e_{18}, e_{19}] &= -e_{18}, [e_{18}, e_{20}] = 3e_{18}, \\
 [e_{18}, e_{21}] &= e_{18}, [e_{18}, e_{22}] = -e_2, [e_{18}, e_{24}] = e_{18}, [e_{18}, e_{25}] = -\frac{e_{14}}{2} + \frac{e_{15}}{2}, [e_{21}, e_{22}] = 2e_{22}, \\
 [e_{21}, e_{23}] &= -2e_{23},
 \end{aligned}$$

$$[e_{22}, e_{23}] = -2e_{21}, [e_{24}, e_{24}] = 2e_{25}, [e_{24}, e_{26}] = -2e_{26}, [e_{25}, e_{26}] = e_{24}.$$

It is a 26-dimensional indecomposable Lie algebra with a non-trivial Levi decomposition. The semi-simple part has two copies of $\mathfrak{sl}(2, \mathbb{R})$ spanned by e_{21}, e_{22}, e_{23} and e_{24}, e_{25}, e_{26} . The solvable part comprises a 20-dimensional where the non-abelian nilradical spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}, e_{17}, e_{18}$ and abelian complement spanned by e_{19}, e_{20} .

$$A_{5,4} : [e_2, e_4] = e_1, [e_3, e_5] = e_1.$$

System of geodesic equations:

$$\ddot{q} = \dot{y}\dot{w} + \dot{x}\dot{z},$$

$$\ddot{x} = 0,$$

$$\ddot{y} = 0,$$

$$\ddot{z} = 0,$$

$$\ddot{w} = 0.$$

Setting $\bar{q} = q - \frac{1}{2}(yw + xz) \Rightarrow \ddot{\bar{q}} = 0$, the system becomes

$$\ddot{\bar{q}} = 0,$$

$$\ddot{x} = 0,$$

$$\ddot{y} = 0,$$

$$\ddot{z} = 0,$$

$$\ddot{w} = 0.$$

The symmetry Lie algebra is $\mathfrak{sl}(7, \mathbb{R})$.

$$A_{5,5} : [e_3, e_4] = e_1, [e_2, e_5] = e_1, [e_3, e_5] = e_2.$$

System of geodesic equations:

$$\ddot{q} = \dot{x}\dot{w} + \dot{y}\dot{z},$$

$$\ddot{x} = \dot{y}\dot{w},$$

$$\ddot{y} = 0,$$

$$\ddot{z} = 0,$$

$$\ddot{w} = 0.$$

Symmetry algebra basis and non-vanishing brackets are, respectively:

$$\begin{aligned}
 e_1 &= D_y, e_2 = D_x, e_3 = D_z, e_4 = tD_q, e_5 = D_q, e_6 = D_z, e_7 = D_w, e_8 = wD_x, e_9 = yD_x, e_{10} \\
 &= zD_q, e_{11} = yD_q, e_{12} = wD_q, \\
 e_{13} &= \frac{w^2D_q}{2} + wD_x, e_{14} = \frac{y^2D_q}{2} + yD_z, e_{15} = \frac{wyD_q}{2} + yD_x, e_{16} = \frac{wyD_z}{2} + wD_z, e_{17} = (x - \frac{wy}{2})D_q, \\
 e_{18} &= \frac{w(w^2 + 3z)D_q}{6} + \frac{w^2D_z}{2} + wD_y, e_{19} = tD_t + \frac{yzD_q}{2} + zD_z - \frac{w(wy - 2x)D_q}{4} + (x - \frac{wy}{2})D_x, \\
 e_{20} &= (\frac{yw^2}{4} - \frac{xw}{2} + q)D_q + \frac{wyD_z}{2} + yD_y, e_{21} = (\frac{w^2y}{4} - zy - \frac{xw}{2} + 2q)D_q + wD_w + \frac{wyD_x}{2}, \\
 e_{22} &= tD_t + \frac{(w^2y - 2xw)D_q}{4} - (x - \frac{wy}{2})D_x, e_{23} = zD_z, e_{24} = \frac{tyD_q}{2} + tD_z, e_{25} = \frac{twD_q}{2} + tD_x, \\
 e_{26} &= (wy - 2x)D_t, e_{27} = \frac{wzD_q}{2} + zD_x, e_{28} = \frac{yzD_q}{2} + zD_z + \frac{(w^2y - 2xw)D_q}{4} - (x - \frac{wy}{2})D_x, \\
 e_{29} &= (\frac{y^2w}{2} - xy)D_q + (wy - 2x)D_z. \\
 [e_1, e_9] &= e_2, [e_1, e_{11}] = e_5, [e_1, e_{14}] = e_{11} + e_6, [e_1, e_{15}] = e_3 + \frac{e_{12}}{2}, [e_1, e_{16}] = \frac{e_{12}}{2}, [e_1, e_{17}] = -\frac{e_{12}}{2}, \\
 [e_1, e_{19}] &= \frac{e_{10}}{2} - \frac{e_{13}}{2}, [e_1, e_{20}] = e_1 + \frac{e_{13}}{2}, [e_1, e_{21}] = -e_{10} + \frac{e_{13}}{2}, [e_1, e_{22}] = \frac{e_{13}}{2}, [e_1, e_{24}] = \frac{e_4}{2}, \\
 [e_1, e_{26}] &= e_8, [e_1, e_{28}] = \frac{e_{10}}{2} + \frac{e_{13}}{2}, [e_1, e_{29}] = e_{16} - e_{17}, [e_2, e_4] = e_5, [e_2, e_{19}] = e_2, \\
 [e_2, e_{22}] &= e_2, \\
 [e_2, e_{24}] &= e_6 + \frac{e_{11}}{2}, [e_2, e_{25}] = e_3 + \frac{e_{12}}{2}, [e_3, e_{17}] = e_5, [e_3, e_{19}] = e_3 + \frac{e_{13}}{2}, [e_3, e_{20}] = -\frac{e_{12}}{2},
 \end{aligned}$$

$$\begin{aligned}
[e_3, e_{21}] &= -\frac{e_{12}}{2}, [e_3, e_{22}] = -\frac{e_{12}}{2} - e_3, [e_3, e_{26}] = -2e_3, [e_3, e_{28}] = -\frac{e_{12}}{2} - e_3, [e_3, e_{28}] = -e_{11} - 2e_6, \\
[e_4, e_8] &= -e_{12}, [e_4, e_{23}] = -e_{10}, [e_4, e_{26}] = 2e_{17}, [e_5, e_{21}] = 2e_5, [e_6, e_{10}] = e_5, \\
[e_6, e_{18}] &= \frac{e_{12}}{2}, [e_6, e_{28}] = e_6 + \frac{e_{11}}{2}, \\
[e_7, e_8] &= e_2, [e_7, e_{12}] = e_5, [e_7, e_{13}] = e_{12} + e_3, [e_7, e_{15}] = \frac{e_{11}}{2}, [e_7, e_{16}] = e_6 + \frac{e_{11}}{2}, \\
[e_7, e_{17}] &= -\frac{e_{11}}{2}, [e_7, e_{18}] = e_{13} + e_1 + \frac{e_{10}}{2}, \\
[e_7, e_{19}] &= -\frac{e_{15}}{2} + \frac{e_{17}}{2}, [e_7, e_{20}] = \frac{e_{15}}{2} - \frac{e_{17}}{2}, [e_7, e_{21}] = -\frac{e_{17}}{2} + e_7 + \frac{e_{15}}{2}, [e_7, e_{22}] = \frac{e_{15}}{2} - \frac{e_{17}}{2}, [e_7, e_{23}] = \frac{e_4}{2} \\
[e_7, e_{26}] &= e_9, [e_7, e_{27}] = \frac{e_{10}}{2}, [e_7, e_{28}] = \frac{e_{15}}{2} - \frac{e_{17}}{2}, [e_7, e_{29}] = e_{14}, [e_8, e_{19}] = e_8, \\
[e_8, e_{21}] &= -e_8, [e_8, e_{22}] = e_8, [e_8, e_{19}] = -e_8, \\
[e_9, e_{19}] &= e_9, [e_9, e_{20}] = -e_9, [e_9, e_{22}] = e_9, [e_9, e_{24}] = e_{14}, [e_9, e_{25}] = e_{15}, [e_{10}, e_{14}] \\
&= -e_{11}, [e_{10}, e_{16}] = -e_{12}, [e_{10}, e_{19}] = -e_{10}, \\
[e_{10}, e_{20}] &= e_{10}, [e_{10}, e_{21}] = 2e_{10}, [e_{10}, e_{24}] = -e_4, [e_{10}, e_{28}] = -e_{10}, [e_{10}, e_{29}] = 2e_{17}, \\
[e_{11}, e_{18}] &= -e_{12}, [e_{11}, e_{21}] = 2e_{11}, [e_{12}, e_{20}] = e_{12}, [e_{12}, e_{21}] = e_{12}, \\
[e_{13}, e_{17}] &= e_{12}, [e_{13}, e_{19}] = e_{13}, [e_{13}, e_{21}] = -e_{13}, [e_{13}, e_{22}] = -e_{13}, [e_{13}, e_{26}] = -2e_8, \\
[e_{13}, e_{28}] &= -e_{13}, [e_{13}, e_{29}] = -2e_{16}, [e_{14}, e_{18}] = -e_{16}, \\
[e_{14}, e_{19}] &= e_{14}, [e_{14}, e_{20}] = -e_{14}, [e_{14}, e_{23}] = e_9, [e_{14}, e_{27}] = e_{15}, [e_{14}, e_{28}] = e_{14}, \\
[e_{15}, e_{17}] &= e_{11}, [e_{15}, e_{18}] = -e_{13}, [e_{15}, e_{19}] = e_{15}, [e_{15}, e_{20}] = -e_{15}, \\
[e_{15}, e_{22}] &= -e_{15}, [e_{15}, e_{26}] = -2e_9, [e_{15}, e_{28}] = -e_{15}, [e_{15}, e_{29}] = -2e_{14}, [e_{16}, e_{19}] = e_{16}, \\
[e_{16}, e_{21}] &= -e_{16}, [e_{16}, e_{23}] = e_8, [e_{16}, e_{27}] = e_{13}, [e_{16}, e_{28}] = e_{16}, \\
[e_{17}, e_{19}] &= -e_{17}, [e_{17}, e_{20}] = e_{17}, [e_{17}, e_{21}] = 2e_{17}, [e_{17}, e_{22}] = e_{17}, [e_{17}, e_{25}] = -e_4, \\
[e_{17}, e_{27}] &= -e_{10}, [e_{17}, e_{28}] = e_{17}, [e_{18}, e_{20}] = e_{18}, \\
[e_{18}, e_{21}] &= -e_{18}, [e_{22}, e_{23}] = -e_{23}, [e_{22}, e_{24}] = e_{24}, [e_{22}, e_{25}] = 2e_{25}, [e_{22}, e_{26}] = -2e_{26}, \\
[e_{22}, e_{27}] &= e_{27}, [e_{22}, e_{29}] = -e_{29}, [e_{23}, e_{24}] = -e_{22} + e_{28}, \\
[e_{23}, e_{25}] &= e_{27}, [e_{23}, e_{28}] = -e_{23}, [e_{23}, e_{29}] = -e_{26}, [e_{24}, e_{26}] = -e_{29}, [e_{24}, e_{27}] = e_{25}, \\
[e_{24}, e_{28}] &= e_{24}, [e_{25}, e_{26}] = -2e_{22}, [e_{25}, e_{28}] = -e_{25}, \\
[e_{25}, e_{29}] &= -2e_{24}, [e_{26}, e_{27}] = 2e_{23}, [e_{26}, e_{28}] = -2e_{27}, [e_{27}, e_{29}] = -2e_{28}, [e_{28}, e_{29}] = -2e_{29}.
\end{aligned}$$

The symmetry algebra is 29-dimensional with a non-trivial Levi decomposition. The semi-simple part is $\mathfrak{sl}(3, \mathbb{R})$ with basis $e_{22}, e_{23}, e_{24}, e_{25}, e_{26}, e_{27}, e_{28}, e_{29}$. The radical is 21-dimensional with an 18-dimensional nilradical spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}, e_{17}, e_{18}$ and abelian complement spanned by e_{19}, e_{20}, e_{21} .

$$A_{5,6} : [e_3, e_4] = e_1, [e_2, e_5] = e_1, [e_3, e_5] = e_2, [e_4, e_5] = e_3.$$

System of geodesic equations:

$$\ddot{q} = 2\dot{x}\dot{w} - z\dot{z}\dot{w},$$

$$\ddot{x} = \dot{y}\dot{w},$$

$$\ddot{y} = \dot{z}\dot{w},$$

$$\ddot{z} = 0,$$

$$\ddot{w} = 0.$$

Symmetry algebra basis and non-vanishing brackets are, respectively:

$$e_1 = D_\rho, e_2 = D_x, e_3 = D_y, e_4 = tD_t, e_5 = D_q, e_6 = D_w, e_7 = zD_q, e_8 = wD_q, e_9 = wD_\rho, e_{10} = zD_\rho,$$

$$e_{11} = twD_q + tD_x, e_{12} = w_2D_q + wD_x, e_{13} = 2xD_q + yD_x + zD_y, e_{14} = wzD_q + zD_x, \\ e_{15} = D_z - \frac{wzD_q}{2},$$

$$e_{16} = (zw - 2y)D_t, e_{17} = \left(\frac{-wz}{2} + y\right)D_q, e_{18} = \frac{w^3}{2}D_q + \frac{w^2}{2}D_x + wD_y,$$

$$e_{19} = (w^2z - 2wy + 4x)D_q + zwD_x + 2zD_y,$$

$$e_{20} = \left(\frac{-zw^2}{2} + yw + \frac{w^4}{12} - 2x\right)D_q + \frac{w^3D_x}{6} + \frac{w^2D_y}{2} + wD_z, e_{21} = tD_t,$$

$$e_{22} = \left(\frac{zw^3}{2} - yw^2 + 2xw + q\right)D_q + wD_w + \left(\frac{w^2z}{2} - wy + 2x\right)D_x + (wz - y)D_y,$$

$$e_{23} = \left(-\frac{zw^3}{6} + \frac{yw^2}{2} - xw + 2q\right)D_q + \left(-\frac{w^2z}{4} + \frac{wy}{2} + x\right)D_x + (2y - \frac{wz}{2})D_y + zD_z.$$

$$[e_1, e_4] = e_5, [e_1, e_{11}] = e_5 + e_8, [e_1, e_{21}] = e_1, [e_2, e_{13}] = 2e_5, [e_2, e_{19}] = 4e_5, [e_2, e_{20}] = -2e_5, [e_2, e_{22}] = 2e_2 + 2e_8,$$

$$[e_2, e_{23}] = e_2 - e_8, [e_3, e_{13}] = e_2, [e_3, e_{16}] = -2e_1, [e_3, e_{17}] = e_5, [e_3, e_{19}] = -2e_8,$$

$$[e_3, e_{20}] = e_8, [e_3, e_{22}] = -e_{12} - e_3,$$

$$[e_3, e_{23}] = \frac{e_{12}}{2} + 2e_3, [e_4, e_9] = -e_8, [e_4, e_{10}] = -e_7, [e_4, e_{16}] = 2e_{17}, [e_4, e_{21}] = -e_4, \\ [e_4, e_{22}] = e_4, [e_4, e_{23}] = 2e_4,$$

$$[e_5, e_{22}] = e_5, [e_5, e_{23}] = 2e_5, [e_6, e_8] = e_5, [e_6, e_9] = e_1, [e_6, e_{11}] = e_4, [e_6, e_{12}] = e_2 + 2e_8,$$

$$[e_6, e_{14}] = e_7, [e_6, e_{15}] = -\frac{e_7}{2},$$

$$[e_6, e_{16}] = e_{10}, [e_6, e_{17}] = -\frac{e_7}{2}, [e_6, e_{18}] = e_{12} + e_3, [e_6, e_{19}] = e_{14} - 2e_{17},$$

$$[e_6, e_{20}] = e_{18} + e_{15} + e_{17}, [e_6, e_{22}] = e_{19} - e_{13} + e_6,$$

$$[e_6, e_{23}] = \frac{e_{13}}{2} - \frac{e_{19}}{2}, [e_7, e_{15}] = -e_5, [e_7, e_{20}] = -e_8, [e_7, e_{22}] = e_7, [e_7, e_{23}] = e_7,$$

$$[e_8, e_{23}] = 2e_8, [e_9, e_{11}] = e_{12}, [e_9, e_{21}] = e_9,$$

$$[e_9, e_{22}] = -e_9, [e_{10}, e_{11}] = e_{14}, [e_{10}, e_{15}] = -e_1, [e_{10}, e_{20}] = -e_9, [e_{10}, e_{21}] = e_{10}, \\ [e_{10}, e_{23}] = -e_{10}, [e_{11}, e_{13}] = 2e_4, [e_{11}, e_{16}] = 2e_{13} - e_{19},$$

$$[e_{11}, e_{19}] = 4e_4, [e_{11}, e_{20}] = -2e_4, [e_{11}, e_{21}] = -e_{11}, [e_{11}, e_{22}] = 2e_{11}, [e_{11}, e_{23}] = e_{11}, \\ [e_{12}, e_{13}] = 2e_8, [e_{12}, e_{19}] = 4e_8, [e_{12}, e_{20}] = -2e_8,$$

$$[e_{12}, e_{22}] = e_{12}, [e_{12}, e_{23}] = e_{12}, [e_{13}, e_{14}] = -2e_7, [e_{13}, e_{15}] = -e_3, [e_{13}, e_{16}] = -2e_{10}, \\ [e_{13}, e_{17}] = e_7, [e_{13}, e_{18}] = -e_{12}, [e_{13}, e_{19}] = -2e_{14} + 4e_{17},$$

$$[e_{13}, e_{20}] = -2e_{17} - e_{18}, [e_{13}, e_{22}] = 3e_{13} - 2e_{19}, [e_{13}, e_{23}] = -e_{13} + e_{19}, [e_{14}, e_{15}] = -e_2 - e_8, \\ [e_{14}, e_{19}] = 4e_7, [e_{14}, e_{20}] = -e_{12} - 2e_7, [e_{14}, e_{22}] = 2e_{14},$$

$$[e_{15}, e_{16}] = e_9, [e_{15}, e_{17}] = -\frac{e_8}{2}, [e_{15}, e_{19}] = 4e_{10}, [e_{16}, e_{21}] = e_{16}, [e_{16}, e_{22}] = e_{16}, \\ [e_{16}, e_{23}] = -2e_{16}, [e_{17}, e_{18}] = -e_8, [e_{17}, e_{19}] = -2e_7,$$

$$[e_{17}, e_{22}] = 2e_{17}, [e_{18}, e_{22}] = -2e_{18}, [e_{18}, e_{23}] = 2e_{18}, [e_{19}, e_{20}] = -2e_{18}, [e_{19}, e_{22}] = -e_{19}, \\ [e_{19}, e_{23}] = e_{19}, [e_{20}, e_{22}] = -e_{20}, [e_{20}, e_{23}] = e_{20}.$$

The symmetry algebra is 23-dimensional solvable. The nilradical is 20-dimensional spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{20}$ and abelian complement spanned by e_{21}, e_{22}, e_{23} .

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