The Generalization of the Stalling’s Theorem

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Abstract

In this paper, we present a relative version of the concept of lower marginal series and give some isomorphisms among $\forall G$-marginal factor groups. Also, we conclude a generalized version of the Stalling’s theorem. Finally, we present a sufficient condition under which the order of the generalized Baer-invariant of a pair of finite groups divides the order of the generalized Baer-invariant of its factor groups.

Keywords: Schur-Baer variety; Pair of groups; $\forall G$-marginal series

Introduction

There exists a long history of interaction between Schur multipliers and other mathematical concepts. This basic notion started by Schur [1], when he introduced multipliers in order to study projective representations of groups. It was known later that the Schur multiplier had a relation with homology and cohomology of groups. In fact, if $G$ is a finite group, then $M(G) \cong H^1(G, \mathbb{C}) \cong H_1(G, \mathbb{Z})$, where $M(G)$ is the Schur multiplier of $G$, $H(G, \mathbb{C})$ is the second cohomology of $G$ with coefficient in $\mathbb{C}$ and $H_1(G, \mathbb{Z})$ is the second internal homology of $G$ [2].

Hopf [3] proved that $M(G) \cong (R \cap F)/[R, F]$. He also proved that the Schur multiplier of $G$ is independent of the free presentation of $G$. Let $(G, N)$ be a pair of groups, where $N$ is a normal subgroup in Ellis [4] defined the Schur multiplier of the pair $(G, N)$ to be the abelian group $M(G, N)$ appearing in the following natural exact sequence

$$H_1(G) \rightarrow H_1(G, N) \rightarrow M(G) \rightarrow M(G, N) \rightarrow 1,$$

where $H_1(\cdot)$ denote the third homology of a group with integer coefficients. He also proved that if the normal subgroup $N$ possess a complement in $G$, then for each free presentation $1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$ of $G$, $M(G, N)$ is isomorphic with the factor group $(R \cap [S, F])/[R, F]$, where $S$ is a normal subgroup of $F$ such that $S \cap R = N$. In particular, if $N = G$ then the Schur multiplier of $(G, N)$ will be $M(G) = (R \cap [S, F])/[R, F]$.

We assume that the reader is familiar with the notions of the verbal subgroup $\forall G$ and the marginal subgroup $\forall G(G)$, associated with a variety of groups $\forall$ and a group $G$ [5] for more information on varieties of groups. Let $F_r$ be the free group freely generated by the countable set $X = \{x_1, x_2, \ldots\}$ and $\forall$ and $V_r$ be two varieties of groups defined by the sets of laws $\forall$ and $V_r$, respectively. Let $N$ be a normal subgroup of a group $G$, then we define $[N^r \forall G]$ to be the subgroup of $G$ generated by the elements of the following set:

$$\{(g_1, g_2, \ldots, g_{r}, v, \ldots, v) \in G \times \forall^r \mid 1 \leq r \leq N, v \in \forall_r, g_1, \ldots, g_r \in G, n \in N\}.$$

It is easily checked that $[N^r \forall G]$ is the least normal subgroup $T$ of $G$ such that $N^rT$ is contained in $V_r(G/T)$ [6].

The first to create the generalization of the Schur multiplier to any variety of groups was Baer [7]. It is well known fact that the recent concept is useful in classifying groups into isologism classes. Leedham-Green and McKay [8] introduced the following generalized version of the Baer-invariant of a group with respect to two varieties $\forall$ and $\forall'$.

Let $G$ be an arbitrary group in $\forall'$ with a free presentation $1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$, in which $F$ is a free group. Clearly, $1 \rightarrow W(G) = W(F)/R$ and hence $W(F) \subseteq R$, therefore,

$$1 \rightarrow R/W(F) \rightarrow F/W(F) \rightarrow G \rightarrow 1$$

is a $\forall'$-free presentation of the group $G$. We call

$$\forall[M(G)] = \frac{R \cap W(F)}{[R, W(F)]} \cong \frac{W(F)}{[W(F), W(F)]}$$

the generalized Baer-invariant of the group $G$. Thus, the Schur multiplier of the pair $(G, N)$ is always abelian and independent of the free presentation of $G$. In particular, if $\forall'$ is the variety of all finite groups and $N = G$ then the generalized Baer-invariant of the pair $(G, N)$ will be

$$\forall[M(G, N)] = \frac{R \cap [S, F]}{[R, [S, F]]} = \forall M(G),$$

which is the usual Baer-invariant of $G$ with respect to $\mathbb{V}$ [8].

It is interesting to know the connection between the Baer-invariant of a pair of finite groups $(G, N)$ and its factor groups with respect to the Schur-Baer variety $\forall'$. In the next section, we show that under some circumstances there are some isomorphisms among $\forall'$-marginal factor groups (Theorem 2.2). Also, a sufficient condition will be given such that the order of the generalized Baer-invariant of a pair of finite groups divides the order of the generalized Baer-invariant of the pair of its factor groups (Theorem 2.5).

Variety $\forall'$ is called a Schur-Baer variety if for any group $G$ in which...
the marginal factor group $G/V(G)$ is finite, then the verbal subgroup $V(G)$ is also finite. Schur [9] proved that the variety of abelian groups is a Schur–Baer variety and Baer [10] showed that a variety defined by outer commutator words carries this property. In 2002, Moghaddam et al. [11] proved that for a finite group $G$, $\text{V}(G)$ is finite with respect to a Schur–Baer variety $V$. In the following lemma we prove similar result for $\text{V}(G, M)$ and $\text{V}(G)$ using another technique.

**Lemma 1.1.** Let $V$ be a Schur–Baer variety and $G$ be a finite group in $V$ with a normal subgroup $N$. Then there exists a group $H$ with a normal subgroup $K$ such that $[K/V(H)] < \infty$.

In particular, $|V(G)| = |V(H)| < \infty$.

**Proof.** Let $G = F/R$ be a free presentation for the group $G$ and $S$ be a normal subgroup of the free group $F$ such that $N \cong S/R$, then

$$R \subseteq \frac{F}{W(F)[R/V(R)]}.$$ 

Let $H = F/V(F)[R/V(R)]$ and $K = S/F(W(F)[R/V(R)])$, then $W(F)/H \cong [V(F)]/N$. But

$$[K/V(H)] = \frac{W(F)[S/V(F)]}{W(F)[R/V(R)]} = \frac{W(F)(K \cap [S/V(F)])}{W(F)(K \cap [S/V(F)])}.$$ 

Also, $[N/V(G)] = \frac{[S/V(F)]}{W(F)(K \cap [S/V(F)])}$. Thus the result holds.

**Stallings’ Theorem**

In the following lemma we present some exact sequences for the generalized Baer-invariant of a pair of groups and its factor groups.

**Lemma 2.1.** Let $G$ be a group with a free presentation $1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$ and $S, T$ be normal subgroups of the free group $F$ such that $T \subseteq S$, $S/R \cong N$ and $T/R \cong K$. Then the following sequences are exact:

(i) \[ 1 \rightarrow \frac{W(F)R}{W(F)\{R/V(R)\}} \rightarrow \text{V}(G, N) \rightarrow \text{V}(G/K, N/K) \rightarrow \text{V}(G[K]/V(G[K]) \rightarrow 1; \]

(ii) $\text{V}(G, N) \rightarrow \text{V}(G/K, N/K) \rightarrow \text{V}(N/[N/V(G)]) \rightarrow \text{V}(N/K) \rightarrow 1$;

(iii) Moreover, if $K$ is contained in $V(G)$, then the following sequence is exact:

\[ 1 \rightarrow W(F)(R \cap [S/V(F)]) \rightarrow \text{V}(G/K, N/K) \rightarrow \text{V}(N/[N/V(G)) \rightarrow 1. \]

**Proof.** Considering the definition mentioned above we can conclude:

\[ \text{V}(G/K, N/K) = \frac{W(F)(R \cap [S/V(F)])}{K \cap [S/V(F)]}, \]

\[ \text{V}(N/[N/V(G))] = \frac{W(F)(R \cap [S/V(F)])}{W(F)[R/V(R)]}. \]

Now one can easily check that the sequences (i) and (ii) are exact.

(iii) Using the assumption, we have $W(F)[V(F)] \subseteq R$. Therefore, one can easily check that the following sequence is exact:

\[ 1 \rightarrow \frac{R \cap [S/V(F)]}{W(F)[R/V(R)]} \rightarrow \text{V}(G/K, N/K) \rightarrow \text{V}(N/[N/V(G)) \rightarrow 1. \]

Let $N$ be a normal subgroup of a group $G$. Then we define a series of normal subgroups of $N$ as follows:

$N = V_s(N, G) \supseteq V_{s-1}(N, G) \supseteq \cdots \supseteq V_1(N, G) \supseteq \cdots$, 

where $V_s(N, G) = [V_s(N, G), V_s(N, G)]$ for all $s \geq 1$. We call such a series the lower $V_s$-marginal series of $N$ in $G$. One may also define the upper $V_s$-marginal series as in studies of Moghaddam et al. [11].

We say that the normal subgroup $N$ of a group $G$ is $V_s$-nilpotent if it has a finite lower $V_s$-marginal series. The shortest length of such series is called the class of $V_s$-nilpotency of $N$ in $G$. If $N = G$, then this is called the lower $V_s$-marginal series of $G$. The group $G$ is said to be $V_s$-nilpotent iff $V_s(G) = 1$, for some positive integer $s$.

Now, we want to show that under some circumstances there are some isomorphisms among $V_s$-marginal factor groups. By using Lemma 2.1, we have the following Theorem, which generalizes 7.9.1 of literature of Hilton and Stammbach [13].

**Theorem 2.2.** Let $f : G \rightarrow H$ be a group homomorphism and $N$ be a normal subgroup of $G$ and $K$ be a normal subgroup of $H$ such that $f(N) \subseteq K$. Suppose $f$ induces isomorphisms $f_1 : G/N \rightarrow H/K$ and $\tilde{f} : N/V(N) \rightarrow K/V(K)$, and that $f \cdot : \text{V}(G, N) \rightarrow \text{V}(H, K)$ is an epimorphism. Then $f$ induces isomorphisms $f_1 : G/V(G, N) \rightarrow H/V(K, H)$ and $\tilde{f} : N/V(N) \rightarrow K/V(K, H)$ for all $n \geq 0$.

**Proof.** At first, we want to mention a point that for making it easier to draw the following diagrams, we would like to introduce $P_n = f_1(N, G)$ and $Q_n = f_1(K, H)$. We proceed by induction. For $n = 0$ the assertion is trivial. For $n = 1$, consider the following diagram:

\[ 1 \rightarrow N/[N/V(G)] \rightarrow G/[V'/G] \rightarrow \text{V}(G) \rightarrow 1 \]

By the hypothesis $\tilde{f}_1$ and $f_1$ are isomorphism. Assume that $n \geq 2$. By considering Lemma 2.1(ii), we can conclude the following commutative diagram:

\[ \text{V}(G, N) \rightarrow \text{V}(G/K, N/K) \rightarrow \text{V}(N/[N/V(G)]) \rightarrow \text{V}(N/K) \rightarrow 1. \]

Note that the naturality of the map $f$ induces isomorphisms $f_1 \in \{1, 2, \ldots, 5\}$ such that $(\ast)$ is commutative. By hypothesis $a_1$ is an epimorphism and $a_2$, $a_3$ are isomorphisms. Also, by considering the induction hypothesis and definition of the Baer-invariant of the pair of groups, $a_3$ is an isomorphism. Hence by five lemma of Rotman’s studies [14] $a_1$ is an isomorphism. Now consider the following diagram and in the same way, $f_1$ is an isomorphism.

Now we obtain the following corollary.
Clearly, which and $H/K$ is an isomorphism, therefore, and $G/P_n$ is an isomorphism. Finally, by the following diagram:

\[
\begin{array}{c}
1 \\
\downarrow \\
N/P_n \\
\downarrow \\
G/N \\
\downarrow \\
1
\end{array}
\]

By the above discussion $\alpha$ is an isomorphism and by induction of hypothesis $\overline{f}_{n+1}$ is an isomorphism, therefore, $\overline{f}_n$ is an isomorphism. Finally, by the following diagram:

\[
\begin{array}{c}
1 \\
\downarrow \\
Q_{n-1}/Q_n \\
\downarrow \\
K/Q_n \\
\downarrow \\
1
\end{array}
\]

And the same way, $f_1$ is an isomorphism.

Now we obtain the following collary.

**Corollary 2.3.** Let $(f,g):(G,N)\rightarrow(H,K)$ are group homomorphisms satisfy the hypotheses of Theorem 2.2. Suppose further that $N$ and $K$ are $V_c$-nilpotent and $V_c$-nilpotent, respectively. Then $f$ and $f_1$ are isomorphisms.

**Proof.** The assertion follows from Theorem 2.2 and the remark that there exists $n \geq 0$ such that $V_n(N,G) = \{1\}$ and $V_n(K,H) = \{1\}$.

Now we have the following theorem, which is a generalization of Stallings’ theorem [15].

**Theorem 2.4.** Let $V$ be a variety of groups and $f: G \rightarrow H$ be an epimorphism. Let $N$ be a $V_c$-nilpotent normal subgroup of $G$ and $K$ be a normal subgroup of $H$ such that $f(N) = K$. If $\ker f \subseteq [W^\gamma G]$ and $WVM(H,K)$ is trivial, then $f$ and $f_1$ are isomorphisms.

**Proof.** Put $M = \ker f$, then $N = [W^\gamma G]$ and $G/N \cong H/K$ and $V_c(N,M) = V_c(K,H)$ for all $n \geq 0$. Now the result follows from Corollary 2.3.

Finally, a sufficient condition will be given such that the order of the generalized Baer-invariant of a pair of finite groups divides the order of the generalized Baer-invariant of the pair of its factor groups with respect to two varieties of groups. Let $\psi: E \rightarrow G$ be an epimorphism such that $\ker \psi \subseteq V^\gamma(E)$. We denote by $(W^\gamma)^\psi(E)$ the intersection of all subgroups of the form $\psi(V^\gamma(E))$. Clearly, $(W^\gamma)^\psi(E)$ is a characteristic subgroup of $G$ which is contained in $V^\gamma(G)$. In particular, if $W$ is the variety of all groups and $V$ is a variety of abelian groups then this subgroup is denoted by $Z^\gamma (G)$ as in literature of Karpilovsky [2].

Now using the above concept we have the following Theorem.

**Theorem 2.5.** Let $K$ be a normal subgroup of $G$ contained in $N \cap (W^\gamma)^\psi(G)$. Then

$$|WVM(G,N)\rangle \text{ divides } |WVM(G/K,N/K)\rangle.$$ 

**Proof.** By theorem 3.2 of Neumann [5], natural homomorphism $WVM(G)\rightarrow WVM(G/K)$ will be a monomorphism. Now the following commutative diagram

\[
\begin{array}{c}
WVM(G,N) \\
\downarrow \\
WVM(G/K,N/K)
\end{array} \subseteq 
\begin{array}{c}
WVM(G) \\
\downarrow \\
WVM(G/K)
\end{array}
\]

implies that the natural homomorphism $WVM(G,N)\rightarrow WVM(G/K,N/K)$ is also a monomorphism. Thus Lemma 1.2 (i) implies that $WVM(G,K)$ is trivial. Now we have $|WVM(G/K,N/K)| = |K \cap [W^\gamma G]| |WVM(G,N)|$, which completes the result.

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