The k-out-of-n System Model with Degradation Facility

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Abstract
In this paper, we study the reliability analysis of k-out-of-n system with degradation facility. Let failure rate, degradable rate and repair rate of components are assumed to be exponentially distributed. There are two types of repair. The first is due to failed state. The second is due to degraded state. The expressions of reliability and mean time to system failure are derived with repair and without repair. We used several cases to analyze graphically the effect of various system parameters on the reliability system.

Keywords: Reliability; Mean time to system failure; Simplex system

Notations
- \( n \): Number of components in the system.
- \( k \): Minimum number of components that must work for the k-out-of-n system to work.
- \( \lambda_1 \): The failure rate of the unit.
- \( \lambda_2 \): The degradable rate of the unit.
- \( \mu_1 \): The repair rate of failed unit.
- \( \mu_2 \): The repair rate of degraded unit.

\( P_{i,j}(t) \): Probability that there are \( i \) degradable units and \( j \) failed units in the system at time \( t \) where \( i=0,1,2,\ldots, n-k+1 \).

\( S \): Laplace transform variable.

\( P_{i,j}(s) \): Laplace transform of \( P_{i,j}(t) \).

\( P_i(t) \): Probability for \( i=0,1,2 \) (0->operable (normal) state; 1->degradable state; 2->failed state).

\( P_i(s) \): Laplace transform of \( P_i(t) \).

\( R_{\text{str}}(t) \): Reliability function of simplex system with degradation.

\( MTTF_{\text{str}} \): Mean time to failure of simplex system with degradation.

\( R_{\text{TMR}}(t) \): Reliability function of TMR system with degradation.

\( MTTF_{\text{TMR}} \): Mean time to failure of TMR system with degradation.

\( R_{(K,N)} \): Reliability of k-out-of-n system or probability that at least \( k \) out of the \( n \) components are working (nonrepairable system), where \( 0 \leq k \leq n \) and both \( k \) and \( n \) are integers.

\( R_{(K,N)} \): Reliability of k-out-of-n system or probability that at least \( k \) out of the \( n \) components are working (repairable system), where \( 0 \leq k \leq n \) and both \( k \) and \( n \) are integers.

\( MTTF_{(K,N)} \): Mean time to failure of a k-out-of-n system.

\( \mu_1 \): The repair rate of failed unit.

\( \mu_2 \): The repair rate of degraded unit.

\( \mu_{1,i} \): Mean repair rate when there are \( n \) failed units in the system.

\( \mu_{2,i} \): Mean repair rate when there are \( n \) degraded units in the system.

Introduction
The general structure of series and parallel systems: the so-called k-out-of-n system. In this type of system, if any combinations of \( k \) units out of \( n \) independent units work, it guarantees the success of the system. For simplicity, assume that all units are identical. Furthermore, all the units in the system are active. The parallel and series systems are special cases of this system for \( k=1 \) and \( k=n \), respectively.

In the past decades; many articles concerning the reliability and availability of standby systems have been published. Among them Galikowsky et al. [1] analyzed the series systems with cold standby components. Wang and Sivazlian [2] examined the reliability characteristics of a multiple-server (M+W) unit system with exponential failure and exponential repair time distributions. Ke et al. [3] studied the reliability measures of a repairable system with warm standby switching failures and reboot delay. Yuge et al. [4] introduced the reliability of a k-out-of-n system with common-cause failures using multivariate exponential distribution. Zhang et al. [5] analyzed the availability and reliability of k-out-of-(M+N): G warm standby systems with two types of failure.

In this paper, we provide a detailed coverage on reliability evaluation of the k-out-of-n systems with degradation. We study the simple model "one unit with one degradable state (simplex)" then we investigate triple modular redundancy (TMR) without repair components. Finally we study the k-out-of-n in details with repair and without repair of components. In addition, we perform numerical results to analyze the effects of the various system parameters on the system reliability.

Non-repairable k-out-of-n System

Simplex system

We get Simplex system when \( n=k=1 \).

Based on the state transition diagram in Figure 1, we can derive the following differential equations:

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Received September 27, 2016; Accepted October 22, 2016; Published October 28, 2016


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From Figure 2, taking LaPlace transforms of the state equations yields:

\[ (s+3\lambda_1)P_{0,0}^*(s)=1 \]  
\[ (s+2\lambda_2)P_{1,0}^*(s) - \lambda_1P_{1,1}^*(s)=0 \]  
\[ (s+2\lambda_3+\lambda_1)P_{0,0}^*(s) - 3\lambda_1P_{0,1}^*(s)=0 \]  
\[ (s+\lambda_1+\lambda_2)P_{1,1}^*(s) - 2\lambda_1P_{1,0}^*(s)-2\lambda_2P_{2,0}^*(s)=0 \]  
\[ (s+\lambda_1+2\lambda_2)P_{2,0}^*(s) - 2\lambda_1P_{1,0}^*(s)=0 \]  
\[ (s+2\lambda_3+2\lambda_2)P_{2,1}^*(s) - \lambda_1P_{1,1}^*(s) - 3\lambda_1P_{2,0}^*(s)=0 \]  
\[ (s)P_{0,1}^*(s) - \lambda_2P_{1,1}^*(s)=0 \]  
\[ (s)P_{1,2}^*(s) - 2\lambda_2P_{2,1}^*(s)=0 \]  

Solving equations (14-21) and taking inverse Laplace transforms of these equations, we get the reliability function of the system

\[ R_{tmr}(t) = \sum_{i=0}^{i=3} P_{ij}(t) \]

\[ R_{tmr}(t) = \frac{3(\lambda_1^2e^{-2\lambda_1 t} + \lambda_2^2e^{-2\lambda_2 t} - 2\lambda_1\lambda_2e^{-(\lambda_1+\lambda_2)t})}{\lambda_1 - \lambda_2} - \frac{2(\lambda_1^2e^{-\lambda_1 t} - \lambda_2^2e^{-\lambda_2 t} + 3\lambda_1\lambda_2^2e^{-(\lambda_1+2\lambda_2)t} - 3\lambda_1^2\lambda_2e^{-(\lambda_1+2\lambda_2)t})}{\lambda_1 - \lambda_2} \]  

As we know the reliability of the triple modular redundancy of a component with one failure rate can be obtained from this equation

\[ R_{sim}^{tmr}(t) = 3R_1^t - 2R_1^t \]

Where \( R_{sim}^{tmr} \) is reliability of a component with failure rate \( \lambda \), from (22) we get

\[ R_{sim}^{tmr}(t) = 3(R_{sim}^{tmr})^3 - 2(R_{sim}^{tmr})^3 \]

The \( MTTF_{tmr} \) can be obtained from this equation

\[ MTTF_{tmr} = \int_0^\infty R_{sim}^{tmr}(t) dt \]

**General system**

We will examine a general model for analysis of such systems when they are nonrepairable. When a k-out-of-n system is put into operation, all n components are in good condition. The system is failed when the number of working components goes down below k or the number of

\[ \frac{\lambda_1}{\lambda_1} + \frac{1}{\lambda_1} \]

**Triple modular redundancy: TMR**

In this scheme, three identical redundant units or modules perform the same task simultaneously with degradable rate. The TMR system only experiences a failure when more than one component fails. In other words, this type of redundancy can tolerate failure of a single component. Figure 2 shows a diagram of the TMR scheme.
failed components has reached \( n-k+1 \). We consider the components in a \( k \)-out-of-\( n \) system are i.i.d. Let the failure rate and degradable rate occur independently of the states of other units and follow exponential distributions with \( \lambda_1, \lambda_2 \) respectively.

At time \( t=0 \) the system starts operation with no failed units. The Laplace transforms of \( P_{i,j}(t) \) are defined by:

\[
\int_0^\infty e^{-st} P_{i,j}(t) \, dt, \quad i=0,1,2,\ldots,n \quad \text{and} \quad j=0,1,2,\ldots,n-k+1.
\]

Based on the model descriptions, the system state transition diagram is given in Figure 3 and it leads to the following Laplace transform expressions for \( P_{i,j}(s) \):

\[
\begin{align*}
(s + n\lambda_1)P_{0,0}(s) &= 1 \quad (26) \\
(s + (n-i)\lambda_1)P_{0,i}(s) - \lambda_1P_{i-1,i}(s) &= 0, 1 \leq i \leq n - kc \quad (27) \\
(s + (n-i)\lambda_1 + i\lambda_2)P_{0,i}(s) - (n-i+1)\lambda_1P_{i,i+1}(s) &= 0, 1 \leq i \leq n - 1 \quad (28) \\
(s + n\lambda_1)P_{0,0}(s) - \lambda_2P_{0,1,0}(s) &= 0 \quad (29) \\
(s + (n-i)\lambda_1)P_{0,i}(s) - (n-i+1)\lambda_1P_{i-1,i}(s) - \lambda_2P_{i,i+1}(s) &= 0, 1 \leq i \leq n - k \quad (30) \\
(s + (n-k-i)\lambda_1)P_{0,i,i-1,i}(s) - (n-k-i)\lambda_1P_{i,i-1,i+1}(s) &= 0, 1 \leq i \leq n - k - 1 \quad (31) \\
(s + (n-k-i+j)\lambda_1 + (k-i+1)\lambda_2)P_{0,i,i-1,i+j}(s) - (k-i+1)\lambda_1P_{i,i-1,i+j+1}(s) &= 0, 1 \leq i \leq n - k \leq j \leq n - k - 1 \quad (32) \\
(s + i\lambda_2)P_{0,0,i}(s) + (n-j-i)\lambda_1P_{i,i+1,j}(s) - (i+2)\lambda_2P_{i,i+1,j+1}(s) - (n-j-i)\lambda_1P_{i-1,i+1,j}(s) &= 0, 0 \leq i \leq n - k - 2, 1 \leq j \leq n - k - 1 \quad (33) \\
(s)P_{i,n-k+1}(s) + (i+1)\lambda_2P_{i,i+1,n-k+1}(s) &= 0, 0 \leq i \leq k - 1 \quad (34)
\end{align*}
\]

From solving equations (26-34) and taking inverse Laplace, we obtain the reliability function as follows:

\[
R_n(k,n) = \lim_{s \to 0} P_{i,j}(s) = \lim_{s \to 0} \left( \sum_{i+j=0}^{n} \sum_{i+j=0}^{n} P_{i,j}(t) \right)
\]

(35)

(36)

The mean time to failure \( MTTF_n(k,n) \) can be obtained from the following relation.

\[
MTTF_n(k,n) = \lim_{s \to 0} P_{i,j}(s) = \lim_{s \to 0} \left( \sum_{i+j=0}^{n} \sum_{i+j=0}^{n} P_{i,j}(s) \right)
\]

(37)

When we perform a sensitivity analysis for changes in the \( R_n(k,n) \) resulting from changes in system parameters \( \lambda_1 \) and \( \lambda_2 \) and. By differentiating equation (36) with respect to \( \lambda_1 \) we obtain,

\[
\frac{\partial R_n(k,n)}{\partial \lambda_1} = \frac{\partial}{\partial \lambda_1} \left( \sum_{i+j=0}^{n} \sum_{i+j=0}^{n} P_{i,j}(t) \right) = \left( \sum_{i+j=0}^{n} \sum_{i+j=0}^{n} \frac{\partial}{\partial \lambda_1} P_{i,j}(t) \right)
\]

(38)

We use the same procedure to get

\[
\frac{\partial R_n(k,n)}{\partial \mu_1} = \frac{\partial}{\partial \mu_1} \left( \sum_{i+j=0}^{n} \sum_{i+j=0}^{n} P_{i,j}(t) \right) = \left( \sum_{i+j=0}^{n} \sum_{i+j=0}^{n} \frac{\partial}{\partial \mu_1} P_{i,j}(t) \right)
\]

We use two cases to study the effect of \( k \) and \( n \) on system reliability.

Case 1: Fix \( \lambda_1 = 0.001, \lambda_2 = 0.008, n=3 \) and choose \( k=1,2,3 \).

Case 2: Fix \( \lambda_1 = 0.001, \lambda_2 = 0.008, k=2 \) and choose \( n=2,3,4 \).

From Figures 4 and 5 we can be observed that the system reliability increases as \( k \) increases or \( n \) increases.

Then, we perform a sensitivity analysis with respect to \( \lambda_1 \) and \( \lambda_2 \). In Figure 6 we can easily observe that the biggest impact almost happened at different time and the order of magnitude of the effect is \((\lambda_1 > \lambda_2)\).

\[
\text{Figure 3: State-transition-rate diagram of non-repairable System with degradation.}
\]
rate that occur independently of the states of other units and follow exponential distributions with \( \lambda_1 \) and \( \lambda_2 \) respectively. In addition, repair rates of failure and degradation are assumed to be exponentially distributed with parameters \( \mu_1 \) and \( \mu_2 \) respectively.

The system starts at time \( t=0 \) and there is no failed or degradable components. When a unit fails, it is immediately sent to the first service line where it is repaired with a time-to-repair that is exponentially distributed with parameter \( \mu_1 \). We have two service lines: the first one repairs failed units and the second one repairs degraded units. When an operating unit degrades, it is repaired with a time-to-repair that is exponentially distributed with parameter \( \mu_2 \) during its working.

We assume that the secession of failure times and repair times are independently distributed random variables. Let us assume that failed units arriving at the repairmen form a single waiting line and are repaired in the order of their breakdowns; i.e., according to the first-come, first-served discipline. Suppose that the repairmen in the two service lines can repair only one failed unit at a time and the repair is independent of the failure of the units. Once a unit is repaired, it is as good as new.

The mean repair rate \( \mu_1 \) is given by:

\[
\mu_1 = \begin{cases} 
  j \mu_0 & \text{if } 1 \leq j \leq n - k \\
  R_j \mu_1 & \text{if } R_j \leq j \leq n - k \\
  0 & \text{otherwise}
\end{cases}
\]

The mean repair rate \( \mu_2 \) is given by:

\[
\mu_2 = \begin{cases} 
  j \mu_2 & \text{if } 1 \leq j \leq n - (R_j - n) \\
  R_j \mu_2 & \text{if } R_j < j \leq n \\
  0 & \text{otherwise}
\end{cases}
\]

Based on the model descriptions, the system state transition diagram is given in Figure 7 and it leads to the following Laplace transform expressions for \( P^n_{R,i}(s) \):

\[
(s + n\lambda_1)P_{R,i}(s) - \mu_1P^n_{R,i+1}(s) - \mu_2P^n_{R,i+2}(s) = s + n\lambda_1
\]

\[
(s + (n-j)\lambda_1 + \mu_1)P_{R,j}(s) - \mu_1P^n_{R,j+1}(s) - \mu_2P^n_{R,j+2}(s) = 0
\]

\[
(s + k\lambda_2 + \mu_2)P^n_{R,k-1}(s) - \lambda_2P^n_{R,k}(s) = 0
\]

\[
(s + (n-i)\lambda_2 + \mu_2 + \mu_1)P^n_{R,i}(s) - (n-i+1)\lambda_1P^{n-1}_{R,i+1}(s) - \mu_1P^n_{R,i+1}(s) - \mu_2P^n_{R,i+2}(s) = 0
\]

\[
(s + n\lambda_2 + \mu_2)P^n_{R,0}(s) - \lambda_2P^n_{R,1}(s) = 0
\]

\[
(s + (n-i)\lambda_2 + \mu_2 + \mu_1)P^n_{R,i}(s) - (n-i+1)\lambda_1P^{n-1}_{R,i+1}(s) - \mu_1P^n_{R,i+1}(s) - \mu_2P^n_{R,i+2}(s) = 0
\]

\[
(s + k\lambda_2 + \mu_2 + \mu_1)P^n_{R,k-2}(s) - \lambda_2P^n_{R,k-1}(s) = 0
\]

\[
(s + (n-i)\lambda_2 + \mu_2 + \mu_1)P^n_{R,i}(s) - (n-i+1)\lambda_1P^{n-1}_{R,i+1}(s) - \mu_1P^n_{R,i+1}(s) - \mu_2P^n_{R,i+2}(s) = 0
\]

\[
(s + n\lambda_2 + \mu_2)P^n_{R,0}(s) - \lambda_2P^n_{R,1}(s) = 0
\]

\[
(s + (n-i)\lambda_2 + \mu_2 + \mu_1)P^n_{R,i}(s) - (n-i+1)\lambda_1P^{n-1}_{R,i+1}(s) - \mu_1P^n_{R,i+1}(s) - \mu_2P^n_{R,i+2}(s) = 0
\]

\[
(s + (n-i)\lambda_2 + \mu_2 + \mu_1)P^n_{R,i}(s) - (n-i+1)\lambda_1P^{n-1}_{R,i+1}(s) - \mu_1P^n_{R,i+1}(s) - \mu_2P^n_{R,i+2}(s) = 0
\]

\[
(s + (n-k-j)i)\lambda_1 + (k-j)\lambda_2 + \mu_1 + \mu_2)P^n_{R,i}(s) - \mu_1P^n_{R,i+1}(s) - \mu_2P^n_{R,i+2}(s) = 0
\]

\[
(s + (n-k-j-i)\lambda_1 + (k-j-i)\lambda_2 + \mu_1 + \mu_2)P^n_{R,i}(s) - \mu_1P^n_{R,i+1}(s) - \mu_2P^n_{R,i+2}(s) = 0
\]

\[
(s + (n-k-j-i)\lambda_1 + (k-j-i)\lambda_2 + \mu_1 + \mu_2)P^n_{R,i}(s) - \mu_1P^n_{R,i+1}(s) - \mu_2P^n_{R,i+2}(s) = 0
\]

\[
(s + (n-k-j-i)\lambda_1 + (k-j-i)\lambda_2 + \mu_1 + \mu_2)P^n_{R,i}(s) - \mu_1P^n_{R,i+1}(s) - \mu_2P^n_{R,i+2}(s) = 0
\]

\[
(s + (n-k-j-i)\lambda_1 + (k-j-i)\lambda_2 + \mu_1 + \mu_2)P^n_{R,i}(s) - \mu_1P^n_{R,i+1}(s) - \mu_2P^n_{R,i+2}(s) = 0
\]

\[
(s + (n-k-j-i)\lambda_1 + (k-j-i)\lambda_2 + \mu_1 + \mu_2)P^n_{R,i}(s) - \mu_1P^n_{R,i+1}(s) - \mu_2P^n_{R,i+2}(s) = 0
\]

\[
(s + (n-k-j-i)\lambda_1 + (k-j-i)\lambda_2 + \mu_1 + \mu_2)P^n_{R,i}(s) - \mu_1P^n_{R,i+1}(s) - \mu_2P^n_{R,i+2}(s) = 0
\]

\[
(s + (n-k-j-i)\lambda_1 + (k-j-i)\lambda_2 + \mu_1 + \mu_2)P^n_{R,i}(s) - \mu_1P^n_{R,i+1}(s) - \mu_2P^n_{R,i+2}(s) = 0
\]
By solving equations (39-49) and taking inverse Laplace transforms (using maple program). We obtain the reliability function as follows:

\[
R_{ij}(t) = \left( \sum_{i+j=0}^{\infty} P_{ij}(s) \right) \cdot L^{-1}
\]

Where \(i, j = 0, 1, 2, \ldots, n \) – \(k\).

The mean time to failure (MTTF) \(R(k, n)\) can be obtained from the following relation.

\[
MTTF_{R}(k, n) = \lim_{s \to 0} \left( \sum_{i+j=0}^{\infty} P_{ij}(s) \right) = \left( \sum_{i+j=0}^{\infty} P_{ij}(0) \right)
\]

We perform a sensitivity analysis for changes in the reliability of the system \(R_{ij}(k, n)\) from changes in system parameters \(\lambda_1, \lambda_2, \mu_1\) and \(\mu_2\) by differentiating equation (50) with respect to \(\lambda\), we obtain

\[
\frac{\partial R_{ij}(k, n)}{\partial \lambda_1} = \frac{\partial}{\partial \lambda_1} \left( \sum_{i+j=0}^{\infty} P_{ij}(t) \right) = \left[ \sum_{i+j=0}^{\infty} \frac{\partial}{\partial \lambda_1} P_{ij}(t) \right]
\]

We use the same procedure to get \(\frac{\partial R_{ij}(k, n)}{\partial \lambda_2}, \frac{\partial R_{ij}(k, n)}{\partial \mu_1}, \frac{\partial R_{ij}(k, n)}{\partial \mu_2}\).

**Numerical results**

In this section, we use MAPLE computer program to provide the numerical results of the effects of various parameters on system reliability and system availability. We choose \(\lambda_1 = 0.001, \lambda_2 = 0.008\) and fix \(\mu_1 = 0.1, \mu_2 = 0.8\). The following cases are analyzed graphically to study the effect of various parameters on system reliability.

Case 1: Fix \(n = 5, k = 2, R_1 = 1, R_2 = 3\).
Case 2: Fix \(n = 5, k = 2, R_1 = 1, R_2 = 1\).

From Figure 8 we find that \(R_1\) don’t effect on system reliability when number of repairman more than one. Figure 9 shows that the repairable system reliability increases as \(R_2\) increases.

Finally we perform sensitivity analysis for system reliability \(R_{ij}(k, n)\) with respect to system parameters.

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**Figure 7:** State-transition-rate diagram of repairable System with degradation facility.

**Figure 8:** Repairable system reliability for different numbers of repairmen in first service line.
1- With respect to all system parameters

From Figures 10 and 11 we can easily observe that the biggest impact almost happened at the same time for $\lambda_2$, $\mu_1$ and $\mu_2$ but it’s happen at shorter time for $\lambda_1$. Moreover, we find $\lambda_1$ is the most prominent parameter while $\lambda_2$, $\mu_1$ and $\mu_2$ are the second, the third and the fourth respectively in magnitude.

2- With respect to $\lambda_1$ at various values of $\lambda_2$

From Figure 12 we can also observe that as $\lambda_2$ decreases the impact of $\lambda_1$ on reliability $R_y(t)$ happened at longer interval time, and the biggest impact almost happened at longer time.

3- With respect to $\lambda_1$ at various values of $\lambda_2$

From Figure 13 we can also observe that as $\lambda_2$ decreases the impact of $\lambda_1$ on reliability $R_y(t)$ happened at longer interval time, and the biggest impact almost happened at longer time, we can observe that the biggest impact of $\lambda_1$ on reliability $R_y(t)$ is not affected by the value of $\lambda_2$ but it’s happen at longer time as $\lambda_2$ decreases.

Conclusions

In this paper, we studied reliability and mean time to system failure for k-out-of-n models with degradation facility. Mathematical model were constructed for these models. Results indicate that the reliability of k-out-of-n nonrepairable system with degradation increases as k or n increases. In repairable system we observe that the order of magnitude of the effect is $(\lambda_1, \lambda_2, \mu_1, \mu_2)$.

References


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