The Normal Ordering Procedure and Coherent State of the Q-Deformed Generalized Heisenberg Algebra

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Abstract

In this paper we discuss the normal ordering procedure of the q-deformed generalized Heisenberg algebra. We also obtain the coherent state for some types of characteristic functions.

Keywords: Heisenberg algebra; q-deformed

Introduction

In the last few years quantum algebras and quantum groups have been the subject of intensive research in several physics and mathematics fields. Quantum groups or q-deformed Lie algebra implies some specific deformation of classical Lie algebra. From the mathematical point of view, it is a non-commutative associative Hopf algebra. The structure and representation theory of quantum groups have been developed extensively by Drinfeld [2] and Jimbo [1]. In the study of the basic hypergeometric function Jackson [3] invented the Jackson derivative and integral, which is now called q-derivative and q-integral. Jackson’s pioneering research enabled theoretical physicists and mathematician to study the new physics or mathematics related to the q-calculus. Much was accomplished in this direction and work is under way to find the meaning of the deformed theory. By using the q-calculus, Arik and Coon [4] proposed the q-deformation of the Heisenberg algebra as follows;

\[ [N, a^\dagger] = a^\dagger, [N, a] = -a \]  

(1)

Where \( N \) is a number operator and \( a = (a^\dagger)^\dagger \). Following the approach of the authors of ref [4], several deformed Heisenberg algebra has been proposed in the literature [5-10]. In most of deformed Heisenberg algebra, authors adopted the same commutation relations between the number operator and step operators and deformed the commutation relation between \( a \) and \( a^\dagger \).

In 2000, the new generalization of the Heisenberg algebra was introduced by Rego-Monteiro and Curado [11-13], which takes the form

\[ H = |N| = \frac{1 - q^N}{1 - q} \]  

(2)

where \( H \) is a hamiltonian of the physical system under consideration and \( f(H) \) is an analytic function of \( H \). For example, if we take

\[ f(H) = 1 + qH \]

then \( H = |N| = \frac{1 - q^N}{1 - q} \) follows immediately.

Q Deformed Generalized Heisenberg Algebra

In this section we discuss the representation theory of the q-deformed generalized Heisenberg algebra. The q-deformed generalized Heisenberg algebra takes the following form;

\[ H a = a^\dagger f(H), \ a H = f(H) a, \ [a, a^\dagger]_N = a^\dagger - qa^\dagger = f(H) - qH, \]  

(3)

where \( H \) is a hamiltonian of the physical system under consideration and \( f(H) \) takes a function which is an analytic function of \( H \). For example, if we take \( f(H) = 1 + qH \), then \( H = |N| = \frac{1 - q^N}{1 - q} \) follows immediately.

The choice of \( f(H) \) gives a lot of deformed algebra, which is the reason why \( f(H) \) is called a characteristic function of the algebra. Now let us introduce the q-Casimir operator as follows;

\[ C_q = a^\dagger a - f(H) = q^{-1} (aa^\dagger - f(H)), \ C_q = C_q \]  

(4)

We demand that the q-Casimir operator obeys

\[ [C_q, H] = 0, \ [a, C_q]_N = 0, \ [C_q, a^\dagger]_N = 0 \]  

(5)

When \( q \) goes to a unity, the q-Casimir operator reduces to an ordinary Casimir operator. If

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We let $C=C_{q}C_{q^{-1}}$, we have
\[ [C, H] = [C, a^\dagger a] = [C, a] = 0 \] (7)

Now let us construct the irreducible representation of the algebra (3) by introducing the ground state $|0\rangle$ with the lowest eigenvalue of $H$ obeying
\[ H|0\rangle = c_{0}|0\rangle \] (8)

Let $|n\rangle$ be a normalized eigenstate of $H$
\[ H|n\rangle = c_{n}|n\rangle, \quad n = 0, 1, 2, \ldots \] (9)

Applying $H$ on $a^\dagger |n\rangle$ yields
\[ H(a^\dagger |n\rangle) = a^\dagger f(H)|n\rangle = f(c_{n})(a^\dagger |n\rangle), \] (10)

Which means that $a^\dagger |n\rangle$ is an eigenstate of $H$ with eigen value $f(c_{n})$. Applying $a$ on the ground state successively, we have
\[ H((a^\dagger)^n |0\rangle) = f^n(c_{0})((a^\dagger)^n |0\rangle), \] (11)

Where $c_{n} = f^n(c_{0})$ denotes the $n$-th iterate of $f$ defined as $f^n(c_{0}) = f(f(f(\ldots f(c_{0})))))$. If we assume that $(a^\dagger)^n |0\rangle$ is proportional to $|n\rangle$, we have
\[ c_{n} = f^n(c_{0}) = f(c_{n-1}) \] (12)

So all eigen values of $H$ are determined from $c_{0}$ through $f$.

Acting $a$ on $|n+1\rangle$ we get
\[ a(H|n+1\rangle) = f(H)(a|n+1\rangle) = c_{n+1}(a|n+1\rangle) = f(c_{n})(a|n+1\rangle). \] (13)

We get
\[ H(a|n+1\rangle) = c_{n}(a|n+1\rangle), \] (14)

Which shows that $a|n+1\rangle$ is also an eigenstate of $H$ with eigen value $c_{n+1}$ and $a|n+1\rangle$ is proportional to $|n\rangle$.

The representation of the q-deformed generalized Heisenberg algebra is then given by
\[ H|n\rangle = c_{n}|n\rangle, \quad n = 0, 1, 2, \ldots \]
\[ a^\dagger |n\rangle = N_{n}|n+1\rangle, \quad a|n\rangle = N_{n-1}|n-1\rangle, \] (15)

Where
\[ N_{n} = f^{n+1}(c_{0}) - q^{n+1}c_{0} = c_{n+1} - q^{n+1}c_{0}, \quad N_{-1} = 0 \] (16)

The relation between step operators and Hamiltonian is given by
\[ aa^\dagger = f(H) - c_{0}P_{q} \] (17)
\[ a^\dagger a = H - q^{-1}c_{0}P_{q}, \] (18)

Where we have
\[ P_{q}|n\rangle = q^{n}|n\rangle \] (19)

Here $P_{q}$ is a q-projection operator satisfying
\[ aP_{q} = qP_{q}a, \quad P_{q}a^\dagger = qa^\dagger P_{q}, \quad P_{q}H = HP_{q} \] (20)

Then we have
\[ |n\rangle = \frac{1}{N_{0}^{\sqrt{n+1}}}(-1)^{n}a^\dagger |0\rangle, \] (21)

Where f-number is defined as
\[ [n]_{f} = \frac{N_{n+1}^{2}}{N_{0}} = \frac{f^{-1}(c_{0}) - q^{n}c_{0}}{f(c_{0}) - q^{-1}c_{0}} \] (22)

And
\[ N_{0}^{2} = f(c_{0}) - q^{-1}c_{0} \] (23)

The $(n+1)$-th eigenvalue $e_{n+1}$ of the Hamiltonian depends on the previous eigenvalue $e_{n}$
\[ e_{n+1} = f^{-1}(e_{n}) \] (24)

So this algebra is sometimes called a one step algebra.

The representation can be rewritten in terms of the f-number as follows;
\[ a^\dagger |n\rangle = N_{n}^{\sqrt{f^{-1}(n) - f^{-1}(n+1)}}, \quad a|n\rangle = N_{n}^{\sqrt{f^{-1}(n) - f^{-1}(n-1)}}, \] (25)

Normal Ordering Process and F-stirling Operator

Now we discuss the normal ordering process for the q-deformed generalized Heisenberg algebra. From the second relation of the eq.(3), we have
\[ a\Phi(H) = \Phi(f(H))a \] (26)

For an arbitrary function $\Phi(H)$. For $\Phi(x) = f(x)$, we get
\[ a^\dagger f(H) = f^{k+1}(H)a^\dagger, \quad f(H)(a^\dagger)^{k} = (a^\dagger)^{k}f^{k+1}(H) \] (27)

Replacing $H \to f^{-1}(H)$ in the first relation of the eq.(3), we have
\[ a^\dagger H = f^{-1}(H)a^{-1} \] (28)

or generally
\[ (a^\dagger)^{k} H = f^{-k}(H)a^{-k} \] (29)

And
\[ H a^{-k} = af^{-k}(H) \] (30)

Where
\[ f^{-k} = f^{-1} \circ f^{-1} \circ \cdots \circ f^{-1} \] (31)

Then we have the following formulas;
\[ a^k (a^\dagger)^{k} = \prod_{j=1}^{k} [f^{-j}(H) - q^{j-1}c_{0}P_{q}] \] (32)
\[ (a^\dagger)^{k} a^{k} = \prod_{j=0}^{k-1} [f^{-j}(H) - q^{-j}c_{0}P_{q}] \] (33)

Where $f(H) = f(H)$.

The f-Stirling operator of the second kind is defined as
The Deformed Heisenberg Algebra Related to the Möbius Transformation

In this section we are going to find the representation for the algebra defined by the relation given in the eq.(3) considering

$$f(H) = \frac{\gamma H + \delta}{\alpha H + \beta},$$

(46)

Where $\alpha, \beta, \gamma, \delta$ are real. The ordinary Heisenberg algebra and q-deformed Heisenberg algebra are obtained from the suitable choice of $\alpha, \beta, \gamma, \delta$ inverse of the Möbius transformation is given by

$$f^{-1}(H) = \frac{\beta H - \delta}{-\alpha H + \gamma},$$

(47)

Where $\beta \gamma - a \delta \neq 0$.

In the choice of the characteristic function given in the eq.(46), the algebra (3) takes the following form;

$$\beta Ha^1 - \gamma a^1 H = \delta a^1 - aHa^1 H,$$

(48)

$$aH - \gamma Ha = \delta a - aHaH,$$

(49)

$$H_a^1 = -q(aH + \beta)a^1 a = \delta + (\gamma - q\beta)H - qaH^2$$

Now let us introduce the following characteristic function;

$$f(H) = \frac{\gamma H}{1 - aH},$$

(50)

Where we assume that $\alpha > 0, 0 < \gamma < 1$. Then we have the following algebra

$$H_a^1 = -\gamma a^1 H = \alpha Ha^1 H,$$

(51)

$$aH - \gamma Ha = \alpha HaH,$$

(52)

The n-the iterate of $f$ is then given by

$$f^n(H) = \frac{\gamma^n H}{1 - \alpha[n]_H},$$

(53)

Where $[n]_H = \frac{1 - \gamma^n}{1 - \gamma}$

Representation

For the characteristic function given in the eq.(49), we have

$$N_0^2 = \frac{c_0(\gamma - q + a\alpha c_0)}{1 - a\alpha c_0}$$

(54)

$$[n]_f = (1 - a\alpha c_0)(\gamma - q + a\alpha c_0[q][n]_f)$$

(55)

$$\frac{1}{\alpha} \frac{\gamma}{q} (1 - \frac{\gamma}{q}) \leq \frac{\alpha}{\gamma} \leq \frac{1}{(1 - \gamma)}$$

(56)
Where we assumed that 
\( \epsilon_0 > 0, \alpha > 0, 0 < \gamma < 1, q > \gamma \)

The representation takes the following form;

\[
H \ket{n} = \left( \frac{\gamma^\epsilon_0}{1-\alpha |n|} \right) |n\rangle
\]

\[
a \ket{n} = \sqrt{c_0(\gamma^n - q^n + \alpha c_0 q^n |n\rangle, [n] - 1)}
\]

\[
a^\dagger \ket{n} = \sqrt{c_0(\gamma^{n+1} - q^{n+1} + \alpha c_0 q^{n+1} |n+1\rangle), [n+1]}
\]

Coherent state

We define a coherent state as a eigenvector of the annihilation operator as follows:

\[
a |z\rangle = z |z\rangle
\]

Where \( z \) is a complex number. The coherent state can be represented by using the number state as follows:

\[
|z\rangle = \sum_{n=0}^{\infty} c_n(z) |n\rangle
\]

Inserting the eq.(58) into the eq.(57), we obtain the following relations:

\[
c_n(z) = \sqrt{\frac{1-\alpha c_0 |n\rangle}{\gamma^n - q^n + \alpha c_0 q^n |n\rangle}} c_{n-1}(z)
\]

Solving the eq.(59), the coherent state is given by

\[
|z\rangle = c_0(|z|^2) \sum_{n=0}^{\infty} \prod_{k=1}^{n} \sqrt{\frac{1-\alpha c_0 |k\rangle}{\gamma^k - q^k + \alpha c_0 q^k |k\rangle}} \left( \frac{z}{\sqrt{c_0}} \right)^n
\]

Where

\[
c_0(|z|^2) = \frac{e_{\gamma,\alpha,\gamma}(z)}{\epsilon_0^{1/2}}
\]

And

\[
e_{\gamma,\alpha,\gamma}(x) = \sum_{n=0}^{\infty} \prod_{k=1}^{n} \sqrt{\frac{1-\alpha c_0 |k\rangle}{\gamma^k - q^k + \alpha c_0 q^k |k\rangle}} \gamma^n
\]

Some Characteristic Functions giving Klauder’s Coherent State

In this section we discuss the some characteristic functions giving q-deformed Klauder’s coherent state (KCS). The KCS should satisfy the normalizability

\[
|z| |z\rangle = 1
\]

and the completeness

\[
\int d_q^2 w(|z|) \langle z | \frac{1}{n+1} | n+1 \rangle = 1
\]

Where we use the q-integral instead of the ordinary integral and assume \( 0 \leq |z|^2 < 1 \).

From the definition of the coherent state

\[
a |z\rangle = z |z\rangle
\]

We have

\[
|z\rangle = c_0(|z|^2) \sum_{n=0}^{\infty} \frac{z^n}{[n+1]} |n\rangle
\]

The case of \( f(x) = \frac{1}{[2]} - qx \)

In this choice we have

\[
f_q(x) = \frac{[n]-q[n-1]x}{[n+1]-q[n]x}
\]

Where we used

\[
q[n+1]-q[n] = [n+2]
\]

If we choose \( \epsilon_0 = 0 \) we have

\[
c_n = N_n = \frac{[n]}{[n+1]}
\]

The coherent state is then given by

\[
|z\rangle = c_0(|z|^2) \sum_{n=0}^{\infty} \sqrt{[n+1]} \gamma^n |n\rangle
\]

Where

\[
c_0^q(x) = (1-x)^2, x = |z|^2
\]

And

\[
(1-\alpha) = \prod_{k=0}^{n-1} (1-q^k a)
\]

Now we have to find the weighting function \( w(x) \) so that the coherent state may satisfy the Completeness. If we set \( z = re^{i\theta}, x = r^2 \)

and assume \( d_q^2 r = rd_q^2 d\theta \), we have

\[
\frac{\pi \sum_{n=0}^{\infty} \delta[n]}{[n]} \int d_q x w(x)(1-x)^2 [n+1]x^n = 1
\]

We can easily find the weighting function

\[
w(x) = \frac{1}{\pi(1-x)^2}
\]

Where we used

\[
\int_0^1 d_q x x^n = \frac{1}{[n+1]}
\]

The case of \( f(x) = \frac{1}{[2]} - qx \)

In this choice we have

\[
f_q(x) = \frac{[n]-q[n-1]\sqrt{x}}{[n+1]-q[n]\sqrt{x}}
\]
If we choose \( c_0 = 0 \) we have
\[
\mathcal{C}_a = N_{a+1}^2 = \left( \frac{[n]}{[n+1]} \right)^2
\]
(77)

The coherent state is then given by
\[
|z\rangle = c_0(|z\rangle \sum_{n=0}^{\infty} [n+1]z^n |n\rangle,
\]
(78)

Where
\[
c_0^2(x) = \frac{(1-x)^3}{1+qx}
\]
(79)

Now we have to find the weighting function \( w(x) \) so that the coherent state may satisfy the completeness, which implies
\[
\sum_{n=0}^{\infty} \left\langle n \right| d_x^a x^n \phi_0(x) \left| n \right\rangle = 1
\]
(80)

We can easily find the weighting function
\[
w(x) = \frac{1-(q)(1+qx)\ln(qx)}{\pi \ln(q)(1-x)}
\]
(81)

Where we used
\[
\int_0^1 d_x x^n \ln(qx) \frac{\ln(1-x)}{(1-q)[n+1]} = 1
\]
(82)

The case of \( f(x) = \left( \frac{[n]-q[n-1]x^{\frac{1}{p}}}{[n+1]-q[n]x^{\frac{1}{p}}} \right)^p \)

In this choice we have
\[
f^*(x) = \left( \frac{[n]-q[n-1]x^{\frac{1}{p}}}{[n+1]-q[n]x^{\frac{1}{p}}} \right)^p
\]
(83)

If we choose \( c_0 = 0 \) we have
\[
\mathcal{C}_a = N_{a+1}^2 = \left( \frac{[n]}{[n+1]} \right)^p
\]
(84)

The coherent state is then given by
\[
|z\rangle = c_0(|z\rangle \sum_{n=0}^{\infty} [n+1]^p z^n |n\rangle,
\]
(85)

Where
\[
c_0^2(x) = \frac{x}{\ln(p)x(x)}
\]
and the q-polylogarithm function is defined by
\[
\text{li}_p^q(x) = \sum_{k=1}^{\infty} [k+n]^p x^{k+1}
\]
(86)

Now we have to find the weighting function \( w(x) \) so that the coherent state may satisfy the completeness, which implies
\[
\sum_{n=0}^{\infty} \left\langle n \right| d_x^a x^n \phi_0(x) c_0^2(x) |n+1\rangle x^n = 1
\]
(87)

The eq.(70) is rewritten as
\[
\int_0^1 d_x x^n = \frac{q^{-1}}{q^{-n+1}}
\]
(89)

If we set \( a = q^{-n} \), we get
\[
\frac{\partial}{\partial a} = (\ln q)\frac{\partial}{\partial a}
\]

Differentiating both sides of the eq.(89) \( p \) times with respect to \( n \), we have
\[
\int_0^1 d_x x^n (\ln x)^p = (1-q)(\ln q)^p \left( a \frac{\partial}{\partial a} \right)^p \left( \frac{1}{1-a} \right)
\]
(90)

Where
\[
\left( a \frac{\partial}{\partial a} \right)^p \left( \frac{1}{1-a} \right) = \left( a \frac{\partial}{\partial a} \right)^p \sum_{k=0}^{n} a^k = \sum_{k=0}^{n} c_k^{(l)} d^k
\]
(91)

We have the following recurrence relation
\[
(k+1) c_{k+1}^{(l)} + (p+1-k) c_k^{(l)} = c_k^{(l-1)} \quad (k = 1, 2, ... , N-1)
\]
(92)

And
\[
c_0^{(l)} = c_0^{(l-1)} = 1
\]
(93)

For \( k = 1 \), the eq.(93) is as follows :
\[
c_1^{(l+1)} = 2c_1^{(l)} + p
\]
(94)

Solving the eq.(95), we have
\[
c_1^{(l)} = \sum_{k=1}^{p} (j-k)c_{j-1}^{(l)} (k+1)^p \theta(j-k-1) \quad (p \geq 2)
\]
(96)

For general \( k \), we have
\[
c_1^{(l)} = \sum_{j=0}^{p} (j-k)c_{j-1}^{(l)} (k+1)^p \theta(j-k-1) \quad (p \geq 2)
\]
(97)

And
\[
c_k^{(l)} = 0 \quad (k \geq 1)
\]
(98)

Where
\[
\theta(x) = \sum_{k=0}^{\infty} \left( \begin{array}{c} l \\vspace{1pt} \\ k \end{array} \right)_{(x)} (x)
\]
(99)

Using the formula (92), we have
\[
\int_0^1 d_x x^n (\ln x)^p = \frac{(-1)^p (q-1)}{(a-1)^{n+1}} \sum_{k=0}^{n} c_k^{(l)} a^{k+1}
\]
(100)

If we expand the summation of the eq.(100) in terms of \( a \), we get
\[
\sum_{k=0}^{n} c_k^{(l)} a^{k+1} = \sum_{i=0}^{n} B_i^{(p)} (a-1)^i
\]
(101)

Where
\[
B_i^{(p)} = \frac{1}{i!} \sum_{k=0}^{p} k(k-1)(k-2)...(k-(l-1))c_k^{(p)}
\]
(102)
Therefore we have the following formula;

\[
\frac{1}{d_x} x^n (\ln x)^r = (-1)^r (\ln q)^n \sum_{p=0}^n \frac{B_p^{(n)} (q-1)^{n-p}}{[n+1]^{n+p}}
\]  

(103)

If we define the new function

\[
L_q(x) = \frac{(1-q)^n}{B_p^{(n)} (\ln q)^n} ((\ln x)^r - (\ln q)^n \sum_{p=0}^n (1-q)^{n-p} B_p^{(n)} L_{p+1,q}(x))
\]  

(104)

We have

\[
\frac{1}{d_x} x^n L_q(x) = \frac{1}{[n+1]^{n+p}}
\]  

(105)

The first few \(L_p(x)\) are as follows;

\[
L_0(x) = 1
\]

\[
L_1(x) = \frac{1-q}{\ln q} (\ln x + \ln q)
\]

(106)

\[
L_2(x) = \frac{(1-q)^2}{4(\ln q)^2} ((\ln x)^2 - 7(\ln q)(\ln x) - 4(\ln q)^2)
\]

Therefore we obtain the weighting function as follows;

\[
w(x) = \frac{\lim_{q \to 1} x^n L_q(x)}{\pi x}
\]  

(107)

Conclusion

In this paper we discussed the normal ordering procedure of the q-deformed generalized Heisenberg algebra, where we introduced the q-deformed generalized Stirling operator of the second kind instead of the Stirling number of the second kind and constructed its generating function. We also discussed the q-deformed generalized Heisenberg algebra whose characteristic function is a M"obius transformation. Finally we discussed some types of characteristic functions giving a Klauder’s coherent state. In fact, it is possible to construct more general algebra as

\[
H_a^+ = a^f(H), \quad aH = f(H)a,
\]

\[
[a,a^+] = \prod_{k=0}^{\infty} (f(H) + k) - q \prod_{k=0}^{\infty} (H + k)
\]  

(108)

It is tempting to investigate, as we did in this paper, the above algebra for some interesting characteristic function. We hope that this topic and its related one will become clear in the near future.

References