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The Studying of Random Cauchy Convection Diffusion Models under Mean Square and Mean Fourth Calculus

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Abstract

The random partial differential equations have a wide range of physical, chemical, and biological applications. The finite difference method offers an attractively simple approximations for these equations. In this paper, the finite difference technique is performed in order to find an approximation solutions for the linear one dimensional convection-diffusion equation with random variable coefficient. We study the consistency and stability of the finite difference scheme under mean square sense. A statistical measure such as mean for the numerical approximation, and the exact solution based on different statistical distributions is computed.

Keywords: Random convection-diffusion equation; Finite difference method; Mean square calculus; Mean fourth calculus

Introduction

The convection-diffusion equation is a parabolic partial differential equation combining the diffusion equation and the advection equation, which describes physical phenomena where particles or energy (or other physical quantities) are transferred inside a physical system due to two processes: diffusion and convection. In its simplest form (when the diffusion coefficient and the convection velocity are constant and there are no sources or sinks).

We can see that the convection-diffusion model in a membrane containing pores or channels lined with positive fixed charges acts as a barrier between intracellular and extracellular compartments filled with electrolyte solutions. The external salt concentration is greater than the internal concentration, thus making it possible to associate the action of the salt solution with sodium. The reason for choosing positive fixed charges in the channels is that this assumption leads to a conductance increase with membrane depolarization. A potential difference E applied across the membrane creates a convectional flow (i.e., bulk flow) with the linear velocity (volume flow per unit of membrane area) by the process of electro osmosis, in the presence of fixed charges, whose density is considered to be relatively low. A pressure difference *P* across the membrane may also be present to influence the volume flow [1-3].

The pollutants solute transport from a source through a medium of air or water is described by a partial differential equation of parabolic type derived on the principle of conservation of mass, and is known as advection-diffusion equation (ADE). In one-dimension it contains two coefficients, one represents the diffusion parameter and the second represents the velocity of the advection of the medium like air or water. In case of porous medium, like aquifer, velocity satisfies the Darcy law and in non-porous medium, like air it satisfies the laminar conditions. The dispersive property differs from pollutant to pollutant.

In water, a pollutant may enter the groundwater zone directly to a landfill site from an industrial site such as nuclear power plants, chemical industries, construction industries etc., and mathematical modelling of the transport of salinity, pollutants and suspended matter in shallow waters involves the numerical solution of a convection-diffusion equation when a pollutant on the surface of a narrow channel. The main purpose in the Pollutants Transport Model is to describe the evolution of the concentration of the pollutant. There are three types for pollutant water surface waters, groundwater, point-source pollution, non point-source pollution and transponder pollution.

Many forms of atmospheric pollution affect human health and the environment at levels from local to global. These contaminants are emitted from diverse sources, and some of them react together to form new compounds in the air. Industrialized nations have made important progress toward controlling some pollutants in recent decades, but air quality is much worse in many developing countries, and global circulation patterns can transport some types of pollution rapidly around the world [4-8]. To complete this model we need to assign the physically relevant boundary and the initial condition. There are three types of boundary conditions:

- The Dirichlet type in which the concentration of the pollutant is prescribed on the boundary.
- The Neumann type flux condition in which the concentration flux normal to the boundary is prescribed.
- The mixed type in which the concentration between the boundary and outside medium.

The initial conditions to be prescribed are generally expressed in terms of background concentration. Although precise background concentration is normally not available, one can consider arbitrary functional form in terms of spatial coordinates.

The performance of this paper is studying the mean square consistency, stability and convergence for one scheme of finite difference method in solving the following linear random convection diffusion equation

$$\begin{cases} u_t + \beta u_x = \alpha u_{xx}, & t \in [0,\infty), x \in R, \\ \quad u(x,0) = u_0(x). \end{cases} \tag{1}$$

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where α is a constant, β is a random variable, t is the time variable, x is the space coordinate and u_t, u_x denote the derivatives with respect to t and x, respectively. Additionally, $u_0(x)$ is a deterministic initial data function. The convection-diffusion equation is one of the most popular equations in physics. Usually, it arises often e.g., in pollutant transport, to describe time and space variations in the motion of particle. Natural phenomena laws are usually enough to derive this equation. Random models can be also used to derive this equation, with the worthy advantage that they provide us with informations about the statistical properties for activity of particle [9,10].

The deterministic convection-diffusion equation reflects two transport mechanisms, the convection and the diffusion [11]. The convection is a kind of heat transfer, it takes place in liquids and gases only because liquids and gases have a physical moving. Additionally, the convection transfers the large mass of particles from a hot part of a fluid rises to a cooler part sinks [12,13]. The diffusion happens when,a single particle of a fluid moving from a higher concentration area to a lower concentration area [14]. For example, in Thongmoon and McKibbin [15], the author has dealt with the deterministic case of this problem numerically by using cubic splines and two standard finite difference schemes. In the same context, the exponential B-spline functions are used for the Galerkin numerical solution of the advection-diffusion equation, also, the extended B-spline are used for the Collection numerical solution of the advection-diffusion equation [16,17], respectively. On the other hand, the random convection diffusion equation happens when the concentration field be under uncertain inputs arises from random flow (velocity) transport or with source (forcing) term. Bishehniasar and Soheili [18], a compact finite difference approximation and semi-Millstream scheme are used in solving the one dimensional advection-diffusion equation with white noise term. Stochastic explicit finite difference methods are discussed for the one dimensional advection-diffusion equation of Ito type [19]. Also, it is clear in Wan et al. [20], the two dimensional advection-diffusion equation of Ito type is solved by using spectral element method.

Our problem (1) states that at a particular location the rate of change of fish numbers with respect to time is determined by the fish's population dynamics and the fluxes of fish to or from that location by means of advection (the velocity terms) and turbulent diffusion (the diffusivity terms). This equation usually has been applied to the transport of non living entities such as pollutants or salt, although its application to the transport of biota such as plankton is becoming more common.

The relative contribution of each term in the problem (1) must be known if the drift migration itself is to be understood, and each term may have biological and physical components. For the physical process of transport the advective terms represent displacement due to the average currents while the diffusivity terms express the dispersal of fish by ocean turbulence. The computations are relative to the velocity at some selected depth, and if the velocity at that depth is not zero, the calculated current velocity is not a true "over-the ground" speed. Geotropic computations neglect currents driven by wind and other frictional forces, yet these currents may be responsible for the majority of the transport, if the larvae occupy the surface layer, so we can take the velocity as a random variable. Even if the currents are well known, larvae undergoing what are nominally drift migrations may nonetheless act in some way to modify their transport and add a biological component to the velocity vectors in the advection-diffusion equation. Because currents may vary substantially with depth, especially if they are produced by the wind or tides, an important way the fish may modify its drift is by controlling its vertical position [21-23]. All provided examples in which fish larvae that were concentrated in surface waters were transported by (wind-induced) currents.

We can develop this model as a random Cauchy problem for advection diffusion model, if we talk about the drift migration of Fish from position to another by random velocity and random diffusivity along a farm in x direction with unbounded spatial domain also, without forces.

In our work, we focus on the convection-diffusion equation including one random variables since, for instance, $u(x,t)$ denotes the concentration of pollutant at x point and t time. β Refers to the advection velocity (wind speed is random variable) in x direction and α denotes the diffusivity coefficient (diffusion of particle is a constant). It is worth to point out in this paper that, the difficulties is in proving the consistency and stability for the scheme we use under mean square sense, where the solution of the problem depends on the involved random variable in the equation.

Our paper has been partitioned as follows, the next section presents some concepts of mean square sense and functional analysis in $L_2(\Omega)$ space. Section 3, studies the finite difference method for the random convection-diffusion equation. Moreover, discussing how to prove the consistency and stability in mean square for our scheme. Section 4, studies numerical approximation with its statistical mean and standard deviation by introducing a numerical example. Section 5,6 are devoted to conclusion and references, respectively.

Preliminaries

In this section, we present some definitions and some important inequalities that we will use in this paper. A real random variable X defined on the probability space $(\Omega,\mathcal{F},P)$ and satisfying the property that $E\left[|X|^p\right]<\infty$, is called p-order random variable (p–$r.v$) where, $p \geq 1$ nd $E[\ ]$ denotes the expected value operator. If $X\in L_p(\Omega)$, then the L_p norm is defined as $\|X\|_p=\left[E\left[|X|^p\right]\right]^{\frac{1}{p}}$.

Proposition 1: A sequence $\{X_n,n>0\}\in L_2(\Omega)$ is mean square convergent to a random variable $X\in L_p(\Omega)$ if $\lim_{n\to\infty}E\left[|X_n-X|^2\right]=0$.

Some Important Inequalities

- Schwarz's Inequality.

$$E[|XY|]\leqslant[E[X^2]E[Y^2]]^{\frac{1}{2}}\ [24].$$

- Hölder's Inequality

$$E[|XY|]\leqslant[E[X^n]]^{\frac{1}{n}}[E[Y^m]]^{\frac{1}{m}},$$

where $n,m>1,\frac{1}{n}+\frac{1}{m}=1$ [25].

If $X,Y\in L_p(\Omega)$, $q\leq 1$ we have

$$\|XY\|_q\leqslant\|X\|_{2q}\|Y\|_{2q}.$$

- Minkowski's Inequality If $1\leq p<\infty$ and $X,Y\in L_p(\Omega)$, then $X,Y\in L_p(\Omega)$, and $[E[|X+Y|^p]]^{\frac{1}{p}}\leqslant[E[|X|^p]]^{\frac{1}{p}}+[E[|Y|^p]]^{\frac{1}{p}}$.

- Lyapunov's inequality For $1\leq r<s<\infty$, then we have $[E[|X|^r]]^{\frac{1}{r}}\leqslant[E[|X|^s]]^{\frac{1}{s}}$.

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Random Finite Difference Scheme

Firstly, in order to apply the finite difference technique to find the approximation solutions for our problem (1), we will discretize the space and the time by finite increasing sequences as follows, the grid points for the space as, $a=x_0<x_1<x_2<x_3<\ldots<x_k=b$. Also, the time points as, $0=t_0<t_1<t_2<t_3<\ldots<t_n=\infty$. Let we define the grid cells for the space to be $\Delta t=(x_k-x_{k-1})$ for $k\geq 1$ and also, define the time steps to be $\Delta t=(t_n-t_{n-1})$ for $t\geq 1$. Consider $u_k^n=u(k\Delta x,n\Delta t)$ approximates the exact solution for the problem (1) as, $u(x,t)$ at the point $(k\Delta x,n\Delta t)$. To formulate the difference scheme according to the problem (1), we must replace the first and second derivative in (1) by difference formulas as follows:

- First-order forward finite difference approximation to u_t

$$u_t(k\Delta x,n\Delta t)\approx\frac{u_k^{n+1}-u_k^n}{\Delta t}$$

- First-order forward finite difference approximation to u_x

$$u_x(k\Delta x,n\Delta t)\approx\frac{u_{k+1}^n-u_k^n}{\Delta x}$$

- Second-order centered finite difference approximation to u_{xx}

$$u_{xx}(k\Delta x,n\Delta t)\approx\frac{u_{k+1}^n-2u_k^n+u_{k-1}^n}{(\Delta x)^2}$$

By substituting in (1), we get the random difference scheme

$$\begin{cases} u_k^{n+1}=(1+r\beta\Delta x-2r\alpha)u_k^n+(r\alpha-r\beta\Delta x)u_{k+1}^n+r\alpha u_{k-1}^n \\ u_k^0=u_0(k\Delta x)=u_0(x_k) \\ r=\dfrac{\Delta t}{(\Delta x)^2},t_n=n\Delta t and x_k=k\Delta x \end{cases} \qquad (2)$$

Consistency of RFDS (2)

A random finite difference scheme (RFDS) $\mathrm{L}_k^n u_k^n=\mathrm{G}_k^n$ that approximating the random partial differential equation (RPDE) Lv=G is consistent in mean square at time $t=(n+1)\Delta t$, if for any smooth function $\Phi=\Phi(x,t)$, we have in mean square

$$E\left[\left|\left(L\Phi-G\right)_k^n-\left(L_k^n\Phi\left(k\Delta x,n\Delta t\right)-G_k^n\right)\right|^2\right]\to 0, \qquad (3)$$

$\Delta t\to 0$, $\Delta x\to 0$ *and* $(k\Delta x,n\Delta t)\to(x,t)$.

Theorem 1: The RFDS (2) that according to the problem (1) is mean square consistent such that $t\to 0$, $\Delta x\to 0$ and $(k\Delta x,n\Delta t)\to(x,t)$.

Proof

$$L(\Phi)_k^n = \frac{\Phi(k\Delta x,(n+1)\Delta t)-\Phi(k\Delta x,n\Delta t)}{\Delta t}+\beta\frac{\Phi((k+1)\Delta x,n\Delta t)-\Phi(k\Delta x,n\Delta t)}{\Delta x} -\alpha\int_{n\Delta t}^{(n+1)\Delta t}\Phi_{xx}(k\Delta x,s)ds,$$

$$L_k^n\Phi(k\Delta x,n\Delta t) = \frac{\Phi(k\Delta x,(n+1)\Delta t)-\Phi(k\Delta x,n\Delta t)}{\Delta t}+\beta\frac{\Phi((k+1)\Delta x,n\Delta t)-\Phi(k\Delta x,n\Delta t)}{\Delta x} -\alpha\frac{\Phi((k+1)\Delta x,n\Delta t)-2\Phi(k\Delta x,n\Delta t)+\Phi((k-1)\Delta x,n\Delta t)}{(\Delta x)^2}.$$

Then,

$$\frac{\Phi((k+1)\Delta x,n\Delta t)-2\Phi(k\Delta x,n\Delta t)+\Phi((k-1)\Delta x,n\Delta t)}{(\Delta x)^2}=\frac{\partial^2\Phi(k\Delta x,n\Delta t)}{\partial x^2}+\mathcal{O}\left((\Delta x)^2\right).$$

From Taylor expansion, the second derivative

$$\frac{\Phi((k+1)\Delta x,n\Delta t)-2\Phi(k\Delta x,n\Delta t)+\Phi((k-1)\Delta x,n\Delta t)}{(\Delta x)^2}=\frac{\partial^2\Phi(k\Delta x,n\Delta t)}{\partial x^2}+\mathcal{O}\left((\Delta x)^2\right).$$

Then, we have

$$E\left[|L(\Phi)_k^n-L_k^n(\Phi)|^2\right]=E[|\alpha\frac{\partial^2\Phi(k\Delta x,n\Delta t)}{\partial x^2}+\mathcal{O}\left((\Delta x)^2\right)-\alpha\int_{n\Delta t}^{(n+1)\Delta t}\Phi_{xx}(k\Delta x,s)ds|^2].$$

As $\Delta t\to 0$, $\Delta x\to 0$ and $(k\Delta x,n\Delta t)\to(x,t)$,

$$E\left[\left|\left(L\Phi-G\right)_k^n-\left(L_k^n\Phi\left(k\Delta x,n\Delta t\right)-G_k^n\right)\right|^2\right]\to 0.$$

Hence, the RFDS (2) is mean square consistent as Δx, $\Delta t\to 0$ and $(k\Delta x,n\Delta t)\to(x,t)$

Stability of RFDS (2)

A random difference scheme $\mathrm{L}_k^n u_k^n=\mathrm{G}_k^n$ that approximating RPDE Lv=G is mean square stable, if there exist some positive constants ε, δ, non-negative constants η, ξ and u^0 is an initial data such that

$$\mathrm{E}\left[\left|\mathrm{u}^{n+1}\right|^2\right]\leq\eta e^{\xi t}\mathrm{E}\left[\left|\mathrm{u}^0\right|^2\right], \qquad (4)$$

for all, $t=(n+1)\Delta t$, $0<\Delta x\leq\varepsilon$, $0<\Delta t\leq$.

Theorem 2 The RFDS equation (2) that according to the problem (1) is mean square stable under the conditions

1. $\Delta t\to 0$, Δx is fixed.
2. β Has positive random distribution.
3. $\mathrm{E}[|\beta|^4]$ (4th order random variable).
4. u^0 is a deterministic function.

Proof

Since,

$$u_k^{n+1}=(1+r\beta\Delta x-2r\alpha)u_k^n+(r\alpha-r\beta\Delta x)u_{k+1}^n+r\alpha u_{k-1}^n,$$

$$E[|u_k^{n+1}|^2]=E[|(1+r\beta\Delta x-2r\alpha)u_k^n+(r\alpha-r\beta\Delta x)u_{k+1}^n+r\alpha u_{k-1}^n|^2].$$

Also since,

$$E[|X+Y|^2]\leqslant[\sqrt{E(|X|^2)}+\sqrt{E(|Y|^2)}]^2,$$

Then,

$$\begin{aligned} E[|u_k^{n+1}|^2] &\leqslant E[|u_k^n+r\beta(\Delta x)u_k^n-2r\alpha u_k^n|^2] \\ &+2E[|r\alpha u_k^n u_{k+1}^n-r\beta(\Delta x)u_k^n u_{k+1}^n+3r^2\alpha\beta(\Delta x)u_k^n u_{k+1}^n-r^2\beta^2(\Delta x)^2u_k^n u_{k+1}^n-2r^2\alpha^2u_k^n u_{k+1}^n|] \\ &+2E[|r\alpha u_k^n u_{k-1}^n+r^2\alpha\beta(\Delta x)u_k^n u_{k-1}^n-2r^2\alpha^2u_k^n u_{k-1}^n|] \\ &+2E[|r^2\alpha^2u_{k+1}^n u_{k-1}^n-r^2\alpha\beta(\Delta x)u_{k+1}^n u_{k-1}^n|] \\ &+E[|r\alpha u_{k+1}^n|^2]+2(E[|r\alpha u_{k+1}^n|^2])^{1/2}(E[|r\beta(\Delta x)u_{k+1}^n|^2])^{1/2} \\ &+E[|r\beta(\Delta x)u_{k+1}^n|^2]+E[|r\alpha u_{k-1}^n|^2]. \end{aligned}$$

Since,

$$E[|X+Y+Z|]\leqslant E[|X|]+E[|Y|]+E[|Z|],$$

Then,

$$\begin{aligned} E[|u_k^{n+1}|^2] &\leqslant E[|u_k^n|^2]+2(E[|u_k^n|^2])^{1/2}(E[|r\beta(\Delta x)u_k^n-2r\alpha u_k^n|^2])^{1/2}+E[|r\beta(\Delta x)u_k^n|^2] \\ &+2(E[|r\beta(\Delta x)u_k^n|^2])^{1/2}(E[|2r\alpha u_k^n|^2])^{1/2}+E[|2r\alpha u_k^n|^2]+2E[|r\alpha u_k^n u_{k+1}^n|] \\ &+2E[|r\beta(\Delta x)u_k^n u_{k+1}^n|]+6E[|r^2\alpha\beta(\Delta x)u_k^n u_{k+1}^n|]+2E[|r^2\beta^2(\Delta x)^2u_k^n u_{k+1}^n|]+4E[|r^2\alpha^2u_k^n u_{k+1}^n|] \\ &+2E[|r\alpha u_k^n u_{k-1}^n|]+2E[|r^2\alpha\beta(\Delta x)u_k^n u_{k-1}^n|]+4E[|r^2\alpha^2u_k^n u_{k-1}^n|] \\ &+2E[|r^2\alpha^2u_{k+1}^n u_{k-1}^n|]+2E[|r^2\alpha\beta(\Delta x)u_{k+1}^n u_{k-1}^n|]+E[|r\alpha u_{k+1}^n|^2] \\ &+2(E[|r\alpha u_{k+1}^n|^2])^{1/2}(E[|r\beta(\Delta x)u_{k+1}^n|^2])^{1/2}+E[|r\beta(\Delta x)u_{k+1}^n|^2]+E[|r\alpha u_{k-1}^n|^2]. \end{aligned}$$

Since,

$$\|X\|_2=[E(X^2)]^{1/2}\quad\forall X\in L_2(\Omega),$$

Then,

$$\begin{aligned} \|u_k^{n+1}\|_2^2 &\leqslant\|u_k^n\|_2^2+2\|u_k^n\|_2\|(r\beta(\Delta x)-2r\alpha)u_k^n\|_2+\|r\beta(\Delta x)u_k^n\|_2^2+2\|r\beta(\Delta x)u_k^n\|_2\|2r\alpha)u_k^n\|_2 \\ &+\|2r\alpha)u_k^n\|_2^2+2r\alpha\|u_k^n\|_2\|u_{k+1}^n\|_2+2r(\Delta x)\|\beta\|_2\|u_k^n\|_2\|u_{k+1}^n\|_2+6r^2\alpha(\Delta x)\|\beta\|_2\|u_k^n\|_2\|u_{k+1}^n\|_2 \\ &+2r^2(\Delta x)^2\|\beta^2\|_2\|u_k^n\|_2\|u_{k+1}^n\|_2+4r^2\alpha^2\|u_k^n\|_2\|u_{k+1}^n\|_2+2r\alpha\|u_k^n\|_2\|u_{k-1}^n\|_2 \\ &+2r^2\alpha(\Delta x)\|\beta\|_2\|u_k^n\|_2\|u_{k-1}^n\|_2+4r^2\alpha^2\|u_k^n\|_2\|u_{k-1}^n\|_2+2r^2\alpha^2\|u_{k+1}^n\|_2\|u_{k-1}^n\|_2 \\ &+2r^2\alpha(\Delta x)\|\beta\|_2\|u_{k+1}^n\|_2\|u_{k-1}^n\|_2+r^2\alpha^2\|u_{k+1}^n\|_2^2+2r^2\alpha(\Delta x)\|u_{k+1}^n\|_2\|\beta u_{k+1}^n\|_2 \\ &+r^2(\Delta x)^2\|\beta u_{k+1}^n\|_2^2+r^2\alpha^2\|u_{k-1}^n\|_2^2. \end{aligned}$$

Since,

$$\|XY\|_2\leqslant\|X\|_4\|Y\|_4\quad\forall X,Y\in L_4(\Omega),$$

Then,

$$\begin{aligned}\| u_k^{n+1} \|_2^2 \leqslant & \| u_k^n \|_4^2 + 2r(\Delta x) \| u_k^n \|_4^2 \| \beta \|_4 + 4r\alpha \| u_k^n \|_4^2 + r^2 (\Delta x)^2 \| \beta \|_4^2 \| u_k^n \|_4^2 \\ & + 4r^2\alpha(\Delta x) \| \beta \|_4 \| u_k^n \|_4^2 + r^2\alpha^2 \| u_k^n \|_4^2 + 2r\alpha \| u_k^n \|_4 \| u_k^{n+1} \|_4 + 2r(\Delta x) \| \beta \|_4 \| u_k^n \|_4 \| u_k^{n+1} \|_4 \\ & + 6r^2\alpha(\Delta x) \| \beta \|_4 \| u_k^n \|_4 \| u_{k+1}^n \|_4 + 2r^2(\Delta x)^2 \| \beta^2 \|_4 \| u_k^n \|_4 \| u_{k+1}^n \|_4 + 4r^2\alpha^2 \| u_k^n \|_4 \| u_{k+1}^n \|_4 \\ & + 2r\alpha \| u_k^n \|_4 \| u_{k-1}^n \|_4 + 2r^2\alpha(\Delta x) \| \beta \|_4 \| u_k^n \|_4 \| u_{k-1}^n \|_4 + 4r^2\alpha^2 \| u_k^n \|_4 \| u_{k-1}^n \|_4 \\ & + 2r^2\alpha^2 \| u_{k+1}^n \|_4 \| u_{k-1}^n \|_4 + 2r^2\alpha(\Delta x) \| \beta \|_4 \| u_{k+1}^n \|_4 \| u_{k-1}^n \|_4 + r^2\alpha^2 \| u_{k+1}^n \|_4^2 + 2r^2\alpha(\Delta x) \| \beta \|_4 \| u_{k+1}^n \|_4^2 \\ & + r^2(\Delta x)^2 \| \beta \|_4^2 \| u_{k+1}^n \|_4^2 + r^2\alpha^2 \| u_{k-1}^n \|_4^2 .\end{aligned}$$

Then,

$$\begin{aligned}\sup_k \| u_k^{n+1} \|_2^2 \leqslant & \sup_k \| u_k^n \|_4^2 + 2r(\Delta x) \| \beta \|_4 \sup_k \| u_k^n \|_4^2 + 4r\alpha \sup_k \| u_k^n \|_4^2 + r^2(\Delta x)^2 \| \beta \|_4^2 \sup_k \| u_k^n \|_4^2 \\ & + 4r^2\alpha(\Delta x) \| \beta \|_4 \sup_k \| u_k^n \|_4^2 + 4r^2\alpha^2 \sup_k \| u_k^n \|_4^2 + 2r\alpha \sup_k \| u_k^n \|_4^2 + 2r(\Delta x) \| \beta \|_4 \sup_k \| u_k^n \|_4^2 \\ & + 6r^2\alpha(\Delta x) \| \beta \|_4 \sup_k \| u_k^n \|_4^2 + 2r^2(\Delta x)^2 \| \beta^2 \|_4 \sup_k \| u_k^n \|_4^2 + 4r^2\alpha^2 \sup_k \| u_k^n \|_4^2 \\ & + 2r\alpha \sup_k \| u_k^n \|_4^2 + 2r^2\alpha(\Delta x) \| \beta \|_4 \sup_k \| u_k^n \|_4^2 \| + 4r^2\alpha^2 \sup_k \| u_k^n \|_4^2 \\ & + 2r^2\alpha^2 \sup_k \| u_k^n \|_4^2 + 2r^2\alpha(\Delta x) \| \beta \|_4 \sup_k \| u_k^n \|_4^2 + r^2\alpha^2 \sup_k \| u_k^n \|_4^2 + 2r^2\alpha(\Delta x) \| \beta \|_4 \sup_k \| u_k^n \|_4^2 \\ & + r^2(\Delta x)^2 \| \beta \|_4^2 \sup_k \| u_k^n \|_4^2 + r^2\alpha^2 \sup_k \| u_k^n \|_4^2 .\end{aligned}$$

Then,

$$\begin{aligned}\sup_k \| u_k^{n+1} \|_2^2 & \leqslant [1 + 8r\alpha + 16r^2\alpha^2 + 4r(\Delta x) \| \beta \|_4 + 4r^2(\Delta x)^2 \| \beta \|_4^2 + 16r^2\alpha(\Delta x) \| \beta \|_4] \sup_k \| u_k^n \|_4^2 \\ & \vdots \\ & \leqslant [1 + 8r\alpha + 16r^2\alpha^2 + 4r(\Delta x) \| \beta \|_4 + 4r^2(\Delta x)^2 \| \beta \|_4^2 + 16r^2\alpha(\Delta x) \| \beta \|_4]^{n+1} \sup_k \| u_k^0 \|_4^2 .\end{aligned}$$

Taking

$$8r\alpha + 16r^2\alpha^2 + 4r(\Delta x) \| \beta \|_4 + 4r^2(\Delta x)^2 \| \beta \|_4^2 + 16r^2\alpha(\Delta x) \| \beta \|_4 \leqslant \lambda^2(\Delta t).$$

Then,

$$\sup_k \| u_k^{n+1} \|_2^2 \leqslant (1 + \lambda^2 \Delta t)^{n+1} \sup_k \| u_k^0 \|_4^2 .$$

Since, u^0 is a deterministic function,

$$\sup_k \| u_k^{n+1} \|_2^2 \leqslant (1 + \lambda^2 \Delta t)^{n+1} \sup_k \| u^0 \|^2 ,$$

and $\Delta t = \dfrac{t}{n+1}$, we have

$$E[| u_k^{n+1} |^2] \leqslant (1 + \frac{\lambda^2 t}{n+1})^{n+1} E[| u^0 |^2] \leqslant e^{\lambda^2 t} E[| u^0 |^2].$$

Hence, the RFDS (2) is mean square stable with $\eta=1$, $\xi=\lambda^2$

Case Studies

We can get random Cauchy problem for the convection diffusion equation in a membrane model if the electro osmosis transportations randomly in the presence of fixed charges acts as a barrier between intracellular component at the initial position and extracellular component at any position along the unbounded domain.

Also, We can get it in a pollutant model if the water streams speed transports randomly, the pollutant diffuse in a deterministic case along the unbounded spatial domain when a pollutant on the surface of an open channel flow, the depth of the water is not constant and also, the turbulent diffusion in the surface of water, channel shapes, channel slop and the nature of the channel material. Turbulence is difficult to define exactly, nevertheless, there are several important characteristics that all turbulent flows possess. These characteristics include unpredictability, rapid diffusivity, high levels of fluctuating velocity, and dissipation of kinetic energy. The velocity at a point in a turbulent flow will appear to an observer to be random or chaotic. The velocity is unpredictable in the sense that knowing the instantaneous velocity at some instant of time is insufficient to predict the velocity a short time later.

In the random case of this model:

The random convection term: The flux is determined by the water stream only but if the velocity of the water stream, the bulk of pollutant that is driven by the stream, without deformation or expansion is random also.

In the deterministic diffusion term:

The pollutant expands from higher concentration regions to lower ones but if the diffusion of pollutant is deterministic, the concentration of pollutant in any region will be deterministic as in this work.

We can also get it in a pollutant model if the wind streams speed transports randomly, the pollutant diffuse in a deterministic case along the unbounded spatial domain when a pollutant of type nitrogen dioxide, ozone, and total suspended particulate matter and carbon monoxide. We can write for example the membrane model as follows.

Let the concentration $u(x,t)$ inside a pore in the membrane, according to our problem, is given by the following partial differential equation

$$\begin{cases} u_t + \beta u_x = \alpha u_{xx}, & t \geq 0, x \in R, \\ u(x,0) = e^{-x^2}, & x \in R \end{cases} \tag{5}$$

Where X is the unbounded space co-ordinate perpendicular to the membrane surfaces, t is the time, the diffusion coefficient is a constant and the advection velocity is a random variable and with an exponential initial condition.

The exact solution

$$u(x,t) = \frac{1}{\sqrt{1+4\alpha t}} e^{-\frac{(x-\beta t)^2}{1+4\alpha t}} . \tag{6}$$

The numerical solution

The random finite difference scheme for this problem is

$$u_k^{n+1} = (1 + r\beta\Delta x - 2r\alpha) u_k^n + (r\alpha - r\beta\Delta x) u_{k+1}^n + r\alpha u_{k-1}^n ,$$

$$u_k^0 = u_0(k\Delta x) = u_0(x_k) = e^{-(k\Delta x)^2},$$

where $r = \dfrac{\Delta t}{(\Delta x)^2}, t_n = n\Delta t$ and $x_k = k\Delta x$.

From the RFDS (2)

$$u_1^1 = (1 + r\beta\Delta x - 2r\alpha) u_1^0 + (r\alpha - r\beta\Delta x) u_2^0 + r\alpha u_0^0 .$$

$$u_1^1 = (1 + r\beta\Delta x - 2r\alpha) e^{-(\Delta x)^2} + (r\alpha - r\beta\Delta x) e^{-(2\Delta x)^2} + r\alpha .$$

$$u_1^2 = (1 + r\beta\Delta x - 2r\alpha) u_1^1 + (r\alpha - r\beta\Delta x) u_2^1 + r\alpha u_0^1 .$$

$$\begin{aligned} u_1^2 & = [(1 + r\beta\Delta x - 2r\alpha)^2 + 2r\alpha(r\alpha - r\beta\Delta x) + (r\alpha)^2] e^{-(\Delta x)^2} \\ & + 2[(r\alpha - r\beta\Delta x)(1 + r\beta\Delta x - 2r\alpha)] e^{-(2\Delta x)^2} + (r\alpha - r\beta\Delta x)^2 e^{-(3\Delta x)^2} \\ & + 2r\alpha(1 + r\beta\Delta x - 2r\alpha). \end{aligned}$$

Verification of the convergence of mean

The ß~Binomial (1.0,0.5), α=1 was explained in Table 1 and ß~ Beta distribution (1.0,2.0), α=1 was explained in Table 2.

The ß~Binomial (1.0,0.5), α=1 was explained in Table 3 and ß~ Beta distribution (1.0,2.0), α=1 was explained in Table 4.

λ^2 for stability

The ß~Binomial distribution (1.0,0.5), α=1 was explained in Table 5 and ß~Beta distribution (1.0,2.0) was explained Table 6. The ß~Exponential (0.5), α=1 was explained in Table 7.

Conclusion

In this work, we have shown that the random finite difference method can be used to obtain the approximation solution stochastic process for the random Cauchy advection diffusion model in one

k	n	x_k	t_n	$E(u(x,t)_{x_k,t_n})$	$E\lvert u_k^n\rvert$	$\frac{\lvert E(u(x,t)_{x_k,t_n}) - E\lvert u_k^n\rvert\rvert}{E(u(x,t)_{x_k,t_n})}$
1	1	0.5	0.01	0.7747527612	0.7753211115	0.00073358925
1	2	0.5	0.005	0.7747527612	0.7752913846	0.00069521972

Table 1: ß~Binomial (1.0,0.5), α=1.

k	n	x_k	t_n	$E(u(x,t)_{x_k,t_n})$	$E\lvert u_k^n\rvert$	$\frac{\lvert E(u(x,t)_{x_k,t_n}) - E\lvert u_k^n\rvert\rvert}{E(u(x,t)_{x_k,t_n})}$
1	1	0.5	0.01	0.7735224945	0.7739513739	0.00055444980
1	2	0.5	0.005	0.7735224945	0.7739421289	0.00054249799

Table 2: ß~ Beta distribution (1.0,2.0), α=1.

k	n	x_k	t_n	$E(u(x,t)_{x_k,t_n})$	$E\lvert u_k^n\rvert$	$\frac{\lvert E(u(x,t)_{x_k,t_n}) - E\lvert u_k^n\rvert\rvert}{E(u(x,t)_{x_k,t_n})}$
1	1	0.5	0.01	0.7768138752	0.7770609473	0.00031805829
1	2	0.5	0.005	0.776813852	0.7770535156	0.00030849140

Table 3: ß~Binomial (1.0,0.5), α=1.

k	n	x_k	t_n	$E(u(x,t)_{x_k,t_n})$	$E\lvert u_k^n\rvert$	$\frac{\lvert E(u(x,t)_{x_k,t_n}) - E\lvert u_k^n\rvert\rvert}{E(u(x,t)_{x_k,t_n})}$
1	1	0.5	0.01	0.7761821248	0.7763760781	0.00024988117
1	2	0.5	0.0025	0.7761821248	0.7763737678	0.00024690468

Table 4: ß~Beta distribution (1.0,2.0), α=1.

Δt	0.1	0.05	0.025	0.005	0.0001	0.000001
λ^2	64.168761	19.578549	6.6628165	0.83233003	0.01419545	0.00014145

Table 5: ß~Binomial distribution (1.0,0.5), α=1 and Δx=0.25.

Δt	0.1	0.05	0.025	0.005	0.0001	0.000001
λ^2	59.9415369	18.388637	6.2987860	0.79647193	0.01365934	0.00013613

Table 6: ß~Beta distribution (1.0,2.0), α=1 and Δx=0.25.

Δt	0.1	0.05	0.025	0.005	0.0001	0.000001
λ^2	67.646950	20.554410	6.95993389	0.86122520	0.01462376	0.00014571

Table 7: ß~Exponential (0.5), α=1 and Δx=0.25.

dimension. The study has been conducted through proving the consistency and stability for the random finite difference scheme we used in this paper under mean square calculus. The convection velocity coefficient must be bounded according to the stability condition. The usefulness of applying our technique to deal with this class of problems has been shown through a number of illustrative examples.

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