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Three Examples of Unbounded Energy for t>0

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Abstract

A solution to the 6thmillenium problem, respect to breakdown of Navier-Stokes solutions and the bounded energy. We have proved that there are initial velocities and forces such that there is no physically reasonable solution to the Navier-Stokes equations for, which corresponds to the case (C) of the problem relating to Navier-Stokes equations available on the website of the Clay Institute.

Keywords: Navier-Stokes equations; Continuity equation; Breakdown; Existence; Smoothness; Physically reasonable solutions; Gradient field; Conservative field; Velocity; Pressure; External force; Unbounded energy; Millennium problem; Uniqueness; Non-uniqueness; 15^{th} Problem of Small; Blowup time

Introduction

A way I see to prove the breakdown solutions of Navier-Stokes equations, following the described in [1], refers to the condition of bounded energy, the finiteness of the integral of the squared velocity of the fluid in the whole space.

We can certainly construct solutions for

$$\frac{\partial ui}{\partial t}+\Sigma_{j=1}^{3}u_j\frac{\partial ui}{\partial x_j}=\nu\nabla^2u_i-\frac{\partial p}{\partial x_i}+f_i,1\le i\le 3 \quad (1)$$

That obeys the condition of divergence-free to the velocity (continuity equation to the constant mass density

$$u(x,0)=u^0(x) \quad (2)\text{(Incompressiable Fluids)}$$

And the initial condition

$$u(x,0)=u^0(x) \quad (3)$$

Where $u_i p_i f_i$ are functions of the position $x\in\mathbb{R}^3$ and the time t ≥ 0, t∈. The constant ν≥0 is the viscosity coefficient, p represents the pressure and is the fluid velocity u=(u1,u2,u3), measured in the position x and time t, with $\nabla^2=\nabla.\nabla=\sum_i^3 1\frac{\partial^2}{\partial_i^2}$. The function $f=(f_1,f_2,f_3)$ has the dimension as acceleration or force per mass unit, but we will keep on naming this vector and its components by its generic name of force such as used in [2]. It's the externally applied force to the fluid, for example, gravity.

The functions U0(x) and f(x,t)must obey, respectively,

$|\partial_x^\alpha u^0(x)|\le C_{ak}(1+|x|)^{-K}$ (4) on R^3 for any $\alpha\in\mathbb{N}_0^3$ and K∈ℝ

And

$|\partial_x^\alpha\partial_t^m f(x,t)|\le C_{amK}(1+|x|+t)^{-K}$ (5) on $\mathbb{R}^3\times[0,¥)$, for any $\alpha\in\mathbb{N}_0^3$, m$\mathbb{N}_0$ and K∈ℝ

With $\mathbb{N}_0$={0,1,2,3,…} (derivatives of order zero does not change the value of function), and a solution(p,u) from (1) to be considered physically reasonable must be continuous and have all the derivatives, of infinite orders, also continuous (smooth), i.e.,

$$p,u\in C^\infty(\mathbb{R}^3\times[0,\infty]) \quad (6)$$

Given an initial velocity u^0 of $C^¥$class, divergence-free (Ñ.u^0=0) on $\mathbb{R}^3$ and an external forces field f also $C^¥$ class on [0,¥), we want, for that a solution to be physically reasonable, beyond the validity of (6), that $u(x,t)$does not diverge to |x |→¥ and satisfy the bounded energy condition, i.e.,

$$\int_{\mathbb{R}^3}|u(x,t)|^2\,dx<C \quad (7)$$

We see that every condition above, from (1) to (7), need to be obeyed to get a solution(p,$\underline{u}$) considered physically reasonable, however, to get the breakdown solutions, (1), (2), (3), (6) or (7) could not be satisfied to some t≥0, in some position $x\in\mathbb{R}^3$, still maintaining (4) and (5) validity.

A way to make this situation (breakdown) happens is when (1) have no possible solution to the pressure p(x,t), when the vector field $\varphi:\mathbb{R}^3\times C^¥[0,¥)\to\mathbb{R}^3$ in

$$\nabla_p=\nu\nabla^2u-\frac{\partial u}{\partial t}-(u.\nabla)u+f=\phi \quad (8)$$

is not gradient, not conservative, in at least one $(x,t)\in\mathbb{R}^3\times C[0,¥)$. In this case, to

$\varphi=(\varphi_1,\varphi_2,\varphi_3)$ not to be gradient, it must be

$$\frac{\partial\phi i}{\partial x_j}\ne\frac{\partial\phi_1}{\partial x_i},i\ne j \quad (9)$$

To some pair and time not negative (for details check, for example, Apostol [2], chapter 10).

If we admit, however, that (1) has a possible solution (p,u) and this also obey (2), (3) and (6), the initial condition $u^0(x)$ and are verifies (2) and (4), the external force $f(x,t)$ verifies (5) and both $u^0(x)$ and $f(x,t)$ are $C^¥$ class, we can try get a breakdown solutions in t≥0 violating the condition (7) (boundary energy) , i.e.,, Choosing $u^0(x)$ or u(x,t) that also obey to

$$\int_{\mathbb{R}^3}|u(x,t)|^2\,dx\to\infty\text{, for some t≥0} \quad (10)$$

C) Breakdown solutions of Navier-Stokes on $\mathbb{R}^3$. Take ν>0 and n=3. Then there exist a smooth and divergence-free vector field $u^0(x)$ on $\mathbb{R}^3$ and a smooth external force on $f(x,t)$on $\mathbb{R}^3\times[0,¥)$, satisfying

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C¥ $f(x,t)$

$$|\partial_x^\alpha u^0(x)| \le C_{aK}(1+|x|)^{-K} \text{ on } \mathbb{R}^3, \forall\alpha, K \tag{4'}$$

And

$$|\partial_x^\alpha \partial_t^m f(x,t)| \le C_{amK}(1+|x|+t)^{-K} \tag{10}$$

For which there exist no solutions $(p,u)$of (1), (2), (3), (6), (7) on $R^3\times[0,\infty)$ It's clear to see that we can solve this problem searching valid velocities which the integral of its square in all space $R^3$is infinite, or also, as shown in (8), searching functions φ non gradients, where the pressure won't be considered a potential function to some instant t ≥ 0. We understand that the α,m shown in (4) and (5) just make sense to $|\alpha|,m\in\{1,2,3,4\ldots\}$ and the negatives K can be neglected because it does not limit the value of functions u^0, f and its derivatives when $|x|\to\infty$ or $t\to\infty$, with C_{ak}, $C_{amk}>0$.

The Schwartz Space *S*

The inequation (4) brings implicitly that $u^0(x)$ must belong to the vectorial space of rapidly decreasing functions, which tend to zero for $|x|\to\infty$, known as Schwartz space $S(R^3)$, , named after the French mathematician Laurent Schwartz (1915-2002) which studied it [3]. These functions and its derivatives of all orders are continuous (C^∞) and decrease faster than the inverse of any polynomial, such that

$$\lim_{|x|\to\infty} |x|^k D^\alpha\varphi(x) = 0 \tag{11}$$

for all $\alpha=(\alpha_1\ldots \alpha_n)$ α_i , non negative integer k≥0, and all integer . is a multiindex, with the convention

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1}_{x_1}\ldots\partial^{\alpha_n}_{x_n}}, |\alpha| = \alpha_1+\ldots+\alpha_n, \alpha_i\in\{0,1,2,\ldots\} \tag{12}$$

D^0 is the operator identity, D^α is a differential operator. An example of function of this space is $u(x)=p(x)e^{-|x|^2}$, where $P(x)$ is a polynomial function.

The following properties are valid [4]:

1) $S(R^n)$ is a vector space; it is closed under linear combinations.

2) $S(R^n)$ is an algebra; the product of functions in $S(R^n)$also belongs to $S(R^n)$ this follows from Leibniz' formula for derivatives of products .

3) $S(R^n)$ is closed under multiplication by polinomials, althrough polynomials are not in S.

4) $S(R^n)$ is closed under differentiation.

5) $S(R^n)$ is closed under translations and multiplication by complex exponentials ($e^{ix.\xi}$)

6) Functions are integrable: $\int_{\mathbb{R}^n}|f(x)|dx<\infty$ for $f\in S(R^n)$. This follows from the fact that and, using polar coordinates, $\int_{\mathbb{R}^n}(1+|x|)^{-(n+1)}dx = C\int_0^\infty(1+r)^{-n-1}r^{n-1}dr<\infty$, i.e., the function $|f$ decreases like r^{-2} (and $(1+r)^{-2}$) at infinity and a finite integral is produced.

By $S(R^3)$ definition and previous properties we see that, as , $u^0(x)\in S(R^3)$ then $\int_{\mathbb{R}^n}|u^0(x)|dx \le M(1+|x|^{-4})dx \le C\int_0^\infty(1+r)^{-2}dr<\infty$ and squared $|u^0(x)|$ and M $(1+|x|)^{-4}$ and we come to the inequality $\int_{\mathbb{R}^n}|u^0(x)|^2dx<\infty$, that contradicts (10).

Another way to check this is that the set $S(R^3)$ it is contained in $L^p(R^n)$for all p, $1\le p<\infty$ [5-9], and in particular for $p=2$ and $n=3$ follows the finiteness of $\int_{\mathbb{R}^n}|u^0(x)|^2dx$. Therefore, if the condition (7) is disobeyed, as we propose in this article, will be for t >0, for example, finding some function $u(x,t)$ like $u^0(x)$ $v(x,t)$, $v(x,0)=1$or $u^0(x)+v(x,t)$, $v(x,0)=0$ with $\int_{\mathbb{R}^3}|u(x,t)|^2dx\to\infty$ and $\int_{\mathbb{R}^3}|u(x,t)|^2dx\to\infty$.

Examples

Really, choosing $u^0(x)\in S(R^3)$ and $f(x,t)\in S(\mathbb{R}^3\times[0,¥))$, obeying this way (4) and (5), remembering that we do not need have $u,p\in S(\mathbb{R}^3\times[0,¥))$ [as a solution, but only $u,p\in C^¥(\mathbb{R}^3\times[0,¥))$ then it is possible to build a solution to the velocity like $u(x,t)=u^0(x)$ $e^{-t}+v(t)$, with $v(0)=0$, such that $\int_{\mathbb{R}^3}|u(x,t)|^2dx\to\infty$, because when $\int_{\mathbb{R}^n}[|u^0(x)|^2e^{-t}+2u^0(x).v(t)]dx\ge 0$, for example, when each component of has the same sign of the respective component $v(t)$ of or the product between them is zero or $\int_{\mathbb{R}^3}[|u^0(x)|.v(t)dx\ge 0$, we will have, $\int_{\mathbb{R}^3}|u(x,t)|^2dx \ge \int_{\mathbb{R}^3}|v(t)|^2dx = |v(t)|^2\int_{\mathbb{R}^3}dx\to\infty$ with $v(t)\ne 0$, $t>0$. We must also choose u,u^0 such that $\nabla.u=\nabla.u^0=0$.

In particular, we choose, for $1\le i\le 3$

$$u^0(x) = e^{-(x_1^2+x_2^2+x_3^2)}(x_2, x_3, x_1x_3, -2x_1x_2) \tag{13.1}$$

$$v_i(t) = w(t) = e^{-t}(1-e^{-t}) \tag{13.2}$$

$$u_i(x,t) = u_i^0(x)e^{-t}+v_i(t) \tag{13.3}$$

$$f_i(x,t) = (-u_i^0 + e^{-t}\Sigma_{j=1}^3 u_j^0\frac{\partial u_i^0}{\partial x_j} + \Sigma_{j=1}^3 v_j\frac{\partial u_i^0}{\partial x_j} - \nu\nabla^2 u_i^0)e^{-t} \tag{13.4}$$

which results to $p(x,t)$, as the only unknown dependent variable yet to be determined,

$$\nabla p + \frac{\partial v}{\partial t} = 0 \tag{14}$$

And then

$$p(x,t) = -\frac{dw}{dt}(x_1+x_2+x_3)+\theta(t) \tag{15}$$

The resulting pressure has a general time dependence $\theta(t)$, should be class $C^¥([0,¥))$ and we can assume limited, and diverges at infinity($|x|\to\infty$), but tends to zero at all space with the increased time (unless possibly $\theta(t)$) , due to the factor e^{-t} that appears in the derivative of $w(t)$,

$$\int_{\mathbb{R}^n}[|u^0(x)|.v(t)dx = 0 \tag{16}$$

In this example $\int_{\mathbb{R}^n}[|u^0(x)|.v(t)dx=0$, and so $\int_{\mathbb{R}^3}|u(x,t)|^2dx\to\infty$ for t>0, as we wanted. Simpler it would be to choose $u^0(x)=0$ Interesting to note that there is no discontinuity in velocity, no singularity (divergence: $|u|\to\infty$), however diverges the total kinetic energy in the whole space, $\int_{\mathbb{R}^3}[|u^2(x)|dx\to\infty$, t>0. . We had as input data $u^0\in L^2(R^3)$, f $L^2(\mathbb{R}^3\times[0,¥))$ but the solution $u^0\notin L^2(R^3\times[0,¥))$, as $p\notin L^2(R^3\times[0,¥))$.

Another interesting example, using the same previous initial velocity, but making v explicitly depend on the position coordinates x_1,x_2 in the direction e_1, e_2, besides time t , and be equal to zero in the direction e_3, with $v(x,0)=0,\nabla=0,v\ne 0$ (v not identically zero), and also obeys all the conditions (1) to (6), is, for $1 \le i \le 3$,

$$u^0(x) = e^{-(x_1^2+x_2^2+x_3^2)}(x_2x_3, x_1x_3, -2x_1x_2) \quad (17.1)$$

$$v(x,t) = e^{-t}w(x,t) \quad (17.2)$$

$$w(x,t) = (w_1(x_1, x_2, t), w_2(x_1, x_2, t), 0) \quad (17.3)$$

$$w(x,0) = 0, \nabla.w = 0, w_3 = v_3 = 0, w \neq 0$$

$$u_i(x,t) = u_i^0(x)e^{-t} + v_i(x,t) = [u_i^0(x) + w_i(x,t)]e^{-t} \quad (17.4)$$

$$f_i(x,t) = (-u_i^0 + e^{-t}\Sigma_{j=1}^3[u_j^0\frac{\partial u_i^0}{\partial x_j} + u_j^0\frac{\partial w_i}{\partial x_j} + w_j\frac{\partial u_i^0}{\partial x_j}] - \nu\nabla^2 u_i^0)e^{-t}$$

$$= (-u_i^0 + \Sigma_{j=1}^3[e^{-t}u_j^0\frac{\partial u_i^0}{\partial x_j} + u_j^0\frac{\partial v_i}{\partial x_j} + v_j\frac{\partial u_i^0}{\partial x_j}] - \nu\nabla^2 u_i^0)e^{-t} \quad (17.5)$$

which results for $p(x,t)$ and $v(x,t)$, as unknowns still to be determined,

$$\frac{\partial p}{\partial x_i} + \frac{\partial v_i}{\partial t} + \Sigma_{j=1}^3 v_j\frac{\partial v_i}{\partial x_j} = \nu\nabla^2 v_i \quad (18)$$

the Navier-Stokes equations without external force.

We know that for n=2 the equation (18) has a solution whose existence and uniqueness is already demonstrated [10-13], therefore, we will transform our three-dimensional system (18) in a two-dimensional system in , which offers as solution a pressure p and a velocity, a priori, with spatially two-dimensional domain, i.e., in the variables (x_1, x_2, t). Resolved, by hypothesis, the equation (18) above, with $v(x, 0)$=0, ∇ v=0, but v not identically zero, we add the third spatial coordinate v_3=0 in the definitive solution for $u(x, t)$, spatially three dimensional, in (17.4), and calculate the external force in (17.5). Choosing $v \in S(\mathbb{R}^2 \times [0, \infty))$ or v polinomial, sine, cosine or their sums to be used in (18), we guarantee that $f\,S(\mathbb{R}^3 \times [0, \infty))$, obeying up (5), with $u^0 \in S(\mathbb{R}^3)$, according (4). Making v limited in module (norm in Euclidean space) we make that u not diverge at $|x| \to \infty$, which is a physically reasonable and desirable condition in [1]. Then build a velocity not identically zero, with $v(x, 0)$=0, ∇ v=0, such that it is relatively simple to solve (18), which is limited in module, can (preferably) go to zero at infinity in at least some situations and can be integrated in $\mathbb{R}^2$, it is C^∞ class and satisfies (5).

Equation (18) also admit a general time dependence to the pressure that is of the form (19)

$$p(x,t) = p_1(x_1, x_2, t) + \theta(t), x \in \mathbb{R}^3 \quad (19)$$

i.e., besides the conventional solution p_1 to the pressure of the two-dimensional problem of the Navier-Stokes equations (18) in the independent variables (x_1, x_2, t), add to p a generic parcel θ (t) only dependent on the time and/or a constant as the definitive pressure solution in the original three-dimensional problem, as we have seen in (15).

The infinitude of the total kinetic energy in this second example, occurs due to the integration of a two dimensional function ($|v|^2$ or $|w^2$) not identically zero in the infinite three dimensional space ($\mathbb{R}^3$).

The total kinetic energy of the problem is, for $v=e^{-t}w$,

$$\begin{aligned} \int_{\mathbb{R}^3} |u|^2\,dx &= \int_{\mathbb{R}^3}(e^{-2t}|u^0|^2 + 2e^{-t}u^0.v + |v|^2)dx \\ &= e^{-2t}\int_{\mathbb{R}^3}(|u^0|^2 + 2u^0.w + |w|^2)dx \end{aligned} \quad (20)$$

Although $\int_{\mathbb{R}^3}\left(|u^0|^2 + 2u^0 \cdot w\right)dx$ is finite, by the properties of functions belonging to the Schwartz space and integrable (the case u^0=0 is elementary), the third parcel in (20) will be infinite in ^{3} for v, $w \neq 0$, though the function can converge and be finite in $\mathbb{R}^2$, that is, if $|v|$ is not identically zero and t >0,

$$\int_{\mathbb{R}^3} |v|^2\,dx = \int_{-\infty}^{+\infty}(\int_{\mathbb{R}^2}|v|^2\,dx)dx_3 = C_2\int_{-\infty}^{+\infty}dx_3 \to \infty \quad (21)$$

hence, for strictly positive and finite t,

$$\int_{\mathbb{R}^3} |u|^2\,dx \to \infty, t > 0, v \neq 0 \quad (22)$$

the violation of condition (7).

Let us now solve (18) explicitly, first in the domain $\mathbb{R}^2 \times [0, \infty)$. In the example 3 your domain will be $\mathbb{R}^3 \times [0, \infty)$. We show that a solution of the type

$$v(x_1, x_2, t) = (X(x_1 - x_2)T(t), X(x_1 - x_2)T(t)) \quad (23)$$

with a given pressure such that

$$\frac{\partial p}{\partial x_1} = -\frac{\partial p}{\partial x_2} = aQ(x_1 - x_2)R(t) + b \quad (24)$$

a, b constants, $a \neq 0$, Q a function of the difference of the spatial coordinates, R a function of time, Q, R not identically zero functions, solve (18) and eliminate the non-linear term, in which case T(0)=0 if resolves (17) and the original system (1), (2), (3). X and T not identically zero, of course.

If v_i=v_j=V in (18), we have to the nonlinear terms

$$\Sigma_{j=1}^3 v_j\frac{\partial v_i}{\partial x_j} = \Sigma_{j=1}^3 V\frac{\partial v}{\partial x_j} = V\Sigma_{j=1}^3\frac{\partial v}{\partial x_j} \quad (25)$$

Doing $\sum_{j=1}^3\frac{\partial v}{\partial x_j} = 0$ in (25) eliminates the nonlinear term, equality which is true when the necessary condition incompressible fluid imposed by us, ∇ v=0, is satisfied, i.e.,

$$\Sigma_{i=1}^3\frac{\partial v_i}{\partial x_j} = \Sigma_{j=1}^3\frac{\partial v}{\partial x_j} = 0 \quad (26)$$

Defining V(x,t)=X($\xi(x)$ $T(t)$, with $x \in \mathbb{R}^n$, then

$$\Sigma_{j=1}^n\frac{\partial v}{\partial x_j} = T(t)\Sigma_{j=1}^n\frac{\partial x(\xi(x))}{\partial x_j} = T(t)\Sigma_{j=1}^n X'(\xi)\frac{\partial \xi(x)}{\partial x_j} = T(t)X'(\xi)\Sigma_{j=1}^n\frac{\partial \xi(x)}{\partial x_j} \quad (27)$$

Functions $\xi(x)$ such that $\sum_{j=1}^n\frac{\partial \xi(x)}{\partial x_j} = 0$ then result in $\sum_{j=1}^n\frac{\partial v}{\partial x_j} = 0$, according (27), following the example of $\xi=x_1-x_2$ in spatial dimension n=2, as it was used in (23).

Substituting (24) in (18), already no nonlinear terms $\sum_{j=1}^3 v_j\frac{\partial v_i}{\partial x_j'}$ and for simplicity making a=1, b=0, comes

$$Q(x_1 - x_2)R(t) + \frac{\partial v}{\partial t} = \nu\nabla^2 V \quad (28)$$

With $V=X(x_1-x_2)$ $T(t)$. We thus transform a system of n partial differential equations nonlinear in a single linear partial differential equation.

Defining $\xi=x_1-x_2$, equation (28) becomes

$$Q(\xi)R(t) + X(\xi)\frac{dT}{dt} = \nu T\nabla^2 X(\xi) \quad (29)$$

We want to get a function $T(t)$ such that T (0)=0, in order that in t=0 we have $v(x,0)$=0, according (23). Let us choose, for example, among other possibilities endless,

$$T(t) = (1 - e^{-t})e^{-t} \quad (30)$$

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limited function in range $0 \le T(t) \le 1$, $t \ge 0$, going to zero for $t \to \infty$.

Thus, by (29), with

$$\frac{dT}{dt} = e^{-t}(2e^{-t} - 1) \tag{31}$$

Comes

$$Q(\xi)R(t) + X(\xi)e^{-t}(2e^{-t} - 1) = v(1 - e^{-t})e^{-t}\nabla^2 X(\xi) \tag{32}$$

Defining $Q(\xi)=X(\xi)$ in (32), to separate our equation with the traditional method of separation of variables used in D.P.E. theory,

$$[R(t) + e^{-t}(2e^{-t} - 1)]X(\xi) = v(1 - e^{-t})e^{-t}\nabla^2 X(\xi) \tag{33}$$

The linear partial differential equation (33) may be resolved by some alternative combinations:

$$\begin{cases} R(t) + e^{-t}(2e^{-t} - 1) = \pm v(1 - e^{-t})e^{-t} \\ X(\xi) = \pm \nabla^2 X(\xi) \end{cases} \tag{34}$$

Or

$$\begin{cases} R(t) + e^{-t}(2e^{-t} - 1) = \pm (1 - e^{-t})e^{-t} \\ X(\xi) = \pm v\nabla^2 X(\xi) \end{cases} \tag{35}$$

or more generally, with $v_1\ v_2 = v > 0$, v_1, $v_2 > 0$,

$$\begin{cases} R(t) + e^{-t}(2e^{-t} - 1) = \pm v_1(1 - e^{-t})e^{-t} \\ X(\xi) = \pm v_2\nabla^2 X(\xi) \end{cases} \tag{36}$$

The differential equation of second order in X, depending on which of the signals we use to ±, The differential equation of second order X in the above systems, depending on which of the signals we use to ± leads us to the Helmholtz equation (negative sign) or a moving steady state governed by Schrödinger equation independent of time (positive signal or negative).

Not intending to use any specific boundary condition for $X(\xi)$ and we do make use of series and Fourier integrals, we choose here the negative sign in ± (the option should be the same in both equations system), and let us make be a trigonometric function, sum of sine and cosine in ξ, i.e.,

$$X(\xi) = A\cos(B\xi) + C\,sen(D\xi) \tag{37}$$

With $\xi = x_1 - x_2$ we have

$$\begin{aligned} \nabla^2 X &= \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)[A\cos(B\xi) + C\,sen(D\xi)] \\ &= \frac{\partial^2}{\partial x_1^2}[A\cos(B\xi) + C\,sen(D\xi)] + \frac{\partial^2}{\partial x_2^2}[A\cos(B\xi) + C\,sen(D\xi)] \\ &= -2[AB^2\cos(B\xi) + CD^2\,sen(D\xi)] \end{aligned} \tag{38}$$

From $X(\xi) = -v_2\nabla_2 X(\xi)$ in (36) comes

$$v_2 = \frac{1}{2B^2} = \frac{1}{2D^2}, v_1 = 2B^2 v = 2D^2 v, |B| = |D| \tag{39}$$

whatever the values of A and C (if A=C=0 or B=D=0 we have the trivial and unwanted solution $v(x, t) \equiv 0$)

The solution for $R(t)$ obtained is then, using $v_1 = 2B^2 v$ given in (39) and the negative sign in (36),

$$R(t) = -e^{-t}[2B^2 v(1 - e^{-t}) + 2e^{-t} - 1] \tag{40}$$

Being $R(0)=-1$.

From (23), (30) and (37) comes up as a possible solution, for $x \in \mathbb{R}^3$ and implicitly introducing the third coordinate space $v_3 \equiv 0$ into v, to

$$\begin{aligned} v(x,t) &= X(x_1 - x_2)T(t)(1,1,0) \\ &= [A\cos(B\xi) + C\,sen(\pm B\xi)](1 - e^{-t})e^{-t}(1,1,0) \end{aligned} \tag{41}$$

which as we can see is not actually a single solution for speed, because of the endless possibilities that we had to set the time dependence $T(t)$, as well as the temporal dependence $X(\xi)$,$\xi = x_1 - x_2$, beyond the arbitrary constants in (41). Even without uniqueness of solution, it meets the requirements we expected: it is limited, continuous of class C^∞, equal to zero at the initial time, tends to zero with increasing time, and has divergent null ($\nabla\ v=0$). Furthermore, when used in the expression (17.5) obtained for the external force f, it does not remove to the force the condition that belong to Schwartz space in relation to space and to the time, i.e., $f \in S(\mathbb{R}^3 \times [0,\infty))$, as can be shown of the S properties that we saw in section § 2 above.

The pressure is obtained by integrating (24) with respect to the difference $=x1 - x2$, with $a=1$, $b=0$, $Q(\xi)=X(\xi)$ and $R(t)$ given in (40),

$$\begin{aligned} p(x,t) - p_0(t) &= R(t)\int_{\xi_0}^{\xi} Q(\xi)d\xi \\ &= e^{-t}[2B^2 v(1 - e^{-t}) + 2e^{-t} - 1]S(\xi) \end{aligned} \tag{42}$$

$$S(\xi) = \frac{A}{B}[sen(B\xi) - sen(B\xi_0)] \pm \frac{A}{B}[\cos(\pm B\xi) - \cos(\pm B\xi_0)]$$

where ξ_0 is the surface $\xi=\xi_0$ and where the pressure is p_0 at time . Again we see that this solution is not unique, not only due solely to the function $p_0(t)$ and respective ξ_0, but also because of the arbitrary constants A and B, the signal ,± beyond from the way $R(T)$ and $Q(\xi)$ were obtained, with a certain freedom of possibilities. $p_0(t)$ substitute the function $\theta(t)$ used in (15) and (19), our generic function of time, or a constant, which must be class $C^\infty([0, \infty))$ and we can assume limited.

Completing the main solution (p,u) that we seek to equation (1), we finally have

$$u(x,t) = u^0(x)e^{-t} + v(x,t) \tag{43}$$

with $u^0(x)$ given in (17.1), $v(x,t)$ in (41) and $f(x,t)$ in (17.5).

The velocity (secondary) v we choose makes the velocity (main) u a function with some properties similar to it: u It is limited oscillating, contains a sum of sine and cosine of a difference in spatial coordinates and decays exponentially over time, or does, not belong to a Schwartz space over the position, nor is square integrable (violating so the inequality (7) in $t > 0$), but is continuous of class C^∞ and does not diverge when $|x| \to \infty$. Their behavior in relation to x_1-$x_2$and the divergence of the total kinetic energy obviously not withdraw of $f(x,t)$ the condition to be pertaining to $S(R^3\times[0,\infty))$, equivalent to inequality (5), since it only depends on $u^0(x)$ and $v(x,t)$. We also have $v(x,0)=0, \nabla\cdot v=0$, $v \in C^\infty(R^3\times[0,\infty))$, the validity of (1), (2), (3), (4) and (6), $u(x,0)=u^0(x)$ $u^0 \in S(R^3)$, with $\nabla\cdot u=0$ and $u \in C^\infty(R^3\times[0,\infty))$, as we wanted.

The Non-uniqueness in *n=2* Spatial Dimension

What's with the proofs of uniqueness of solutions of the Navier-Stokes equations in spatial dimension $n=2$?

Not possible to examine all the available proofs, you can at least understand that such proofs should not take into account the absence of the nonlinear term in the Navier-Stokes equations, $\sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} \equiv ((u\cdot\nabla)u)_i$, $1 \le I \le n$, and it was this lack that we use in our second example.

Similarly to this cause, also realize that different equations of the type Navier-Stokes equations with the absence of one or more terms of their complete equation, and which nevertheless have the same initial condition *u(x,0)=u⁰(x),* will probably, in the general case, different solutions *u(x,t)* among them, and so there can be no uniqueness of solution from the full Navier-Stokes equation, with all terms. If all always presented the same and only solution would suffice for us to solve the simplest of them only, for example, $\Delta p=-\frac{\partial u}{\partial t}$ or $\nabla p=\nu\nabla^2 u$ (Poisson Equation if $\nabla p\neq 0$ or by Laplace if $\nabla p=0$) or $\frac{\partial u}{\partial t}=\nu\Delta^2 u$ (Heat Equation with $\nabla p=0$), all with $u(x,0)=u^0(x)$, and check if the sum of the other missing terms is zero to apply the solution obtained in the reduced equation. If so, the solution of the reduced equation is also solution of the complete equation. Important example of this absence are the Euler equations, which differ from the Navier-Stokes equations by the absence of differential operator nabla applied to u, $\nabla^2 u=\nabla u$, due to the viscosity coefficient be zero, $\nu=0$.

It is easy to prove that the above three equations, as well as the equation $\Delta p+\frac{\partial u}{\partial t}=\nu\Delta^2 u$, not may actually have a unique solution, given only the initial condition for velocity $u(x,0)=u^0(x)$. On the contrary, the complete form of the Navier-Stokes equations, where we assume that $\sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} \equiv ((u\cdot\Delta)u)_i \neq 0, 1\leq i\leq n$, It has uniqueness of solution for n=2 and at least a short period of time not null $[0,T]$ for n=3, where T is known as blowup time. Let us add all of these equations the condition of incompressibility, $\nabla\cdot u=0$.

This is so an interesting problem of Combinatorial Analysis applied to Mathematical Analysis and Mathematical Physics.

Uniqueness in *n*=2 Spatial Dimension

We found that the system

$$\begin{cases} \nabla p+\frac{\partial v}{\partial t}=\nu\nabla^2 v \\ (v.\nabla)v=0 \\ \nabla.v=0 \\ v(x,0)=0 \end{cases} \tag{44}$$

has infinite solutions to the velocity of the form

$$v(x,t)=X(\xi)T(t)(1,1), \xi=x_1-x_2 \tag{45}$$

with, $T(0)=0$ however there are known proofs of the uniqueness of

$$\begin{cases} \nabla p+\frac{\partial v}{\partial t}+(v.\nabla)v=\nu\nabla^2 v \\ \nabla.v=0 \\ v(x,0)=0 \end{cases} \tag{46}$$

contradicting what we got.

No need to linger in the known proofs, exposing all its details, repeating his steps, you can be seen in Leray [10], Ladyzhenskaya [11], Leray [14], among others, that the proofs of existence and uniqueness are based on the complete form of the Navier-Stokes equations, for example (46), and not in a dismembered form of the Navier-Stokes equations, as (44).

The Navier-Stokes equations without external force with n=2 are (using $x=x_1$ and $y=x_2$)

$$\begin{cases} \frac{\partial p}{\partial x}+\frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x}+u_1\frac{\partial u_1}{\partial y}=\nu\nabla^2 u_1 \\ \frac{\partial p}{\partial y}+\frac{\partial u_2}{\partial t}+u_1\frac{\partial u_2}{\partial x}+u_2\frac{\partial u_2}{\partial y}=\nu\nabla^2 u_2 \end{cases} \tag{47}$$

We can dispose the system up in a similar form to a system of linear equations,

$$\begin{cases} u_1\frac{\partial u_1}{\partial x}+u_2\frac{\partial u_1}{\partial y}=\nu\nabla^2 u_1-\frac{\partial p}{\partial x}-\frac{\partial u_1}{\partial t} \\ u_1\frac{\partial u_2}{\partial x}+u_2\frac{\partial u_2}{\partial y}=\nu\nabla^2 u_2-\frac{\partial p}{\partial y}-\frac{\partial u_2}{\partial t} \end{cases} \tag{48}$$

and then in the form of a matrix equation,

$$\begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=\begin{pmatrix} \nu\nabla^2 u_1-\frac{\partial p}{\partial x}-\frac{\partial u_1}{\partial t} \\ \nu\nabla^2 u_2-\frac{\partial p}{\partial y}-\frac{\partial u_2}{\partial t} \end{pmatrix} \tag{49}$$

Calling

$$A=\begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} \end{pmatrix} \tag{50}$$

$$U=\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{51}$$

$$B=\begin{pmatrix} \nu\nabla^2 u_1-\frac{\partial p}{\partial x}-\frac{\partial u_1}{\partial t} \\ \nu\nabla^2 u_2-\frac{\partial p}{\partial y}-\frac{\partial u_2}{\partial t} \end{pmatrix} \tag{52}$$

The solution for U of the equation (49), $AU=B$, is

$$U=A^{-1}B \tag{53}$$

That for its existence and unique solution must be

$$\det A=\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y}-\frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}\neq 0 \tag{54}$$

That is,

$$\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y}\neq\frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x} \tag{55}$$

Rule should also be obeyed for t=0 (again can lead us to cases (C) and (D) of [1] applying the method in matrix 3×3, i.e., n=3, however, with appropriate choice of p or $\partial u/\partial t$ the system will be possible).

If we use the condition of incompressibility$\nabla\cdot u=0$,

$$\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}=0 \tag{56}$$

i.e.,

$$\frac{\partial u_1}{\partial x}=-\frac{\partial u_2}{\partial y} \tag{57}$$

becomes the condition (55) in

$$-\left(\frac{\partial u_2}{\partial y}\right)^2\neq\frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x} \tag{58}$$

or equivalently,

$$-(\frac{\partial u_1}{\partial x})^2 \neq \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x} \quad (59)$$

Since this condition must be valid for all t, in t=0 must obey to

$$-(\frac{\partial u_1^0}{\partial x})^2 \neq \frac{\partial u_1^0}{\partial y}\frac{\partial u_2^0}{\partial x} \quad (60)$$

and

$$-(\frac{\partial u_2^0}{\partial x})^2 \neq \frac{\partial u_1^0}{\partial y}\frac{\partial u_2^0}{\partial x} \quad (61)$$

Using $u(x,y,0) = u^0(x,y) = (u_1^0(x,y), u_2^0(x,y))$

If the initial velocity $u^0$is such that either disobeyed (60) or (61) then either there is no solution to the system (47) (impossible system) or there will be a non-unique solution (indeterminate system), as in theory linear systems.

Defining

$$U_1 = \begin{pmatrix} v\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t} & \frac{\partial u_1}{\partial y} \\ v\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t} & \frac{\partial u_2}{\partial y} \end{pmatrix} \quad (62)$$

and

$$U_2 = \begin{pmatrix} \frac{\partial u_1}{\partial x} & v\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t} \\ \frac{\partial u_2}{\partial x} & v\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t} \end{pmatrix} \quad (63)$$

the solution for u_1,$u_2$will be

$$u_1 = \frac{\det U_1}{\det A} \quad (64)$$

and

$$u_2 = \frac{\det U_2}{\det A} \quad (65)$$

Being

$$\det U_1 = (v\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t})\frac{\partial u_2}{\partial y} - (v\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t})\frac{\partial u_1}{\partial y} \quad (66)$$

and

$$\det U_2 = (v\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t})\frac{\partial u_1}{\partial x} - (v\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t})\frac{\partial u_2}{\partial x} \quad (67)$$

with det A given in (54), then we have

$$u_1 = \frac{\det U_1}{\det A} = \frac{(v\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t})\frac{\partial u_2}{\partial y} - (v\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t})\frac{\partial u_1}{\partial y}}{\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y} - \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}} \quad (68)$$

and

$$u_2 = \frac{\det U_2}{\det A} = \frac{(v\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t})\frac{\partial u_1}{\partial x} - (v\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_2}{\partial t})\frac{\partial u_2}{\partial x}}{\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y} - \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}} \quad (69)$$

Using the incompressibility equation in the determinant of A,

$$u_1 = -\frac{(v\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t})\frac{\partial u_2}{\partial y} - (v\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t})\frac{\partial u_1}{\partial y}}{(\frac{\partial u_2}{\partial y})^2 + \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}} \quad (70)$$

and

$$u_2 = -\frac{(v\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t})\frac{\partial u_1}{\partial x} - (v\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t})\frac{\partial u_2}{\partial x}}{(\frac{\partial u_1}{\partial x})^2 + \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}} \quad (71)$$

It is true that the solutions (equations) above are as or more complicated as the original equations (47), and seems there no use whatsoever in resolving them.

But this complicated form can be reached with more certainty to the following conclusion: the Navier-Stokes (and Euler) equations have a symmetry between the variables, both dependent as independent. The same can also be realized directly in (47).

The symmetry in this case of n=2 is

$$u_1 \leftrightarrow u_2 \quad (72.1)$$

$$x \leftrightarrow y \quad (72.2)$$

p and t being unchanged:

$$p \leftrightarrow p \quad (73.1)$$

$$t \leftrightarrow t \quad (73.2)$$

This suggests, if not completely solves, the question of the solution of these equations. If the equations themselves are symmetrical with respect to certain transformations, so we hope that their solutions are also under these transformations. The same method can be applied also for $n \geq 3$, with the rule (e.g.)

$$x_i \mapsto x_{i+1}, x_{n+1} \equiv x_1 \quad (74.1)$$

$$x_i \mapsto x_{i+1}, x_{n+1} \equiv x_1 \quad (74.2)$$

$$p \leftrightarrow p \quad (74.3)$$

$$t \leftrightarrow t \quad (74.4)$$

In this case it is necessary that the initial condition $u(x,0)=u^0(x)$ also obey these symmetries, but remains unchanged the condition of incompressibility:

$$\sum_{i=1}^{n}\frac{\partial u_i}{\partial x_i} = \sum_{i=1}^{n}\frac{\partial u_i^0}{\partial x_i} = 0$$

If we provide u_2 (x,y,t)as input data in our system then we can conclude that the solution for u_1, supposedly symmetrical to by the rule (72) previous, is

$$u_1(x,y,t) = u_2(y,x,t) \quad (75)$$

i.e., we exchange x by y, and vice versa, in the solution previously given for $u_2$and we equate $u_1$to the result of this transformation. Lack get the pressure p or else if it has also been given, check that the variables u_1, u_2, p really satisfy the original system.

The general form of the solution to the pressure p, which must satisfy

$$\nabla p + \frac{\partial u}{\partial t} + (u \cdot \nabla)u = v\nabla^2 u \quad (76)$$

is

$$p - p_0(t) = \int_{(x_0,y_0)}^{(x,y)} [v\nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla)u].dl \quad (77)$$

Where we assume that in the position $(x,y)=(x_0,y_0)$and at the instant t the pressure is equal to $p_0(t)$. The integration occurs in any way

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between (x_0,y_0) and (x,y), because the pressure should be a potential function of the integral of (77) in order that (47) has solution.

It is also expected that p be symmetric with respect to variables x and y, in other words,

$$p(x,y,t) = p(y,x,t) \tag{78}$$

as well as in 3 dimensions, using $x=x_1, y=x_2, z=x_3$,

$$p(x,y,z,t) = p(y,z,x,t) = p(z,x,y,t) \tag{79}$$

Of course (74), (75), (78) and (79) implicitly admits that we have rectangular symmetry in the initial and contour conditions of the system. Since this symmetry does not occur, for example, have another type of symmetry, spherical, cylindrical, or even none symmetry (general case), the equalities (74), (75), (78) and (79) do not need to be met. Thus, the solution for the case that there is none symmetry is still a problem to be solved, assuming that there is at least one solution (when the system is possible; as we said, it can be proved that the system is always possible, for example, with appropriate choice of p or $\partial u/\partial t$).

Finally then developed the foregoing, our example 3, which seeks a unique solution to the Navier-Stokes system with n=3, all terms of the equation, nonzero external force, and provides infinite total kinetic energy to the system (1) to (6) in $t > 0$will be based on the example 2, but again need to resort to the absence of non-linear term in the equation auxiliary with n=3. Since (18), the Navier-Stokes equation without external force, has as initial condition the zero initial velocity, the only possible velocity for your solution with all the terms is also the zero velocity due to the uniqueness of the solutions in the form complete this equation (abstracting constant generic pressures and/or time functions), solution that does not interest us. So, we need again that (18) does not have the non-linear term. The uniqueness of the main equation solution in three dimensions, however, at least in short time, is guaranteed because it contains all terms (again, except for not unique pressure), including the applied external force (which itself depends of not unique solution of the auxiliary equation with n=3).

The third example is a generalization of the example 2, with the velocity components v_2 and v_3 proportional to the component v_1,

$$v_1 = X(\xi)T(t), \xi = x_1 + \frac{1}{\alpha}x_2 - 2\frac{1}{\beta}x_3, \alpha \neq 0, \beta \neq 0 \tag{80.1}$$

$$v_2 = \alpha v_1 \tag{80.2}$$

$$v_3 = \beta v_1 \tag{80.3}$$

α and β non-zero constant. We could also use other coefficients combinations in the variables x_i in ξ, whenever $\nabla\cdot(\xi I)=0$, with $I=(1,1,1)$. In the example 2 we use α=1, β=0.

We will choose the components of the initial velocity u^0 with some property of symmetry. It is not easy to think of not constant velocities with symmetrical components u_1^0 and simultaneously whose divergent $\nabla\cdot u^0$ is null. The velocities with symmetry which the i-th component does not contain the i-th coordinate space, for all i (natural) in $1 \le i \le n$, fulfill this requirement: $\frac{\partial u_1^0}{\partial x_i} = 0$. Alternatively we can use the known vector equality $\nabla\cdot(\nabla\times \boldsymbol{A})=0$, ie, choosing a vector u^0 that has a potential vector $\boldsymbol{A}$, i.e.,$u^0=\nabla\times \boldsymbol{A}$. So we choose primarily a vector $\boldsymbol{A}$ that has the symmetry properties we expect.

Be $\boldsymbol{A}=(A_1,A_2,A_3)$ the potential vector we want. Doing $A_1=A_2=A_3=e^{-r^2}$, with $r^2 = x_1^2 + x_2^2 + x_3^2$, the value assigned to the initial velocity $u^0(x)$ will be

$$u^0(x) = rot\ A = 2e^{-r^2}(-x_2 + x_3, -x_3 + x_1, -x_1 + x_2) \tag{81}$$

Following the equations 17 of Example 2, let us now for $x \in R^3$,

$$v(x,t) = e^{-t}w(x,t) \tag{82.1}$$

$$w(x,t) = (w_1(x_1,x_2,x_3,t), w_2(x_1,x_2,x_3,t), w_3(x_1,x_2,x_3,t)) \tag{82.2}$$

$$w(x,0) = 0, \nabla.w = 0, w \neq 0$$

$$u_i(x,t) = u_i^0(x)e^{-t} + v_i(x,t) = [u_i^0(x) + w_i(x,t)]e^{-t} \tag{82.3}$$

$$\frac{\partial p}{\partial x_i} + \frac{\partial v_i}{\partial t} + \sum_{j=1}^{3} v_j \frac{\partial v_i}{\partial x_j} = \nu\nabla^2 v_i \tag{82.4}$$

Which results for $p(x,t)$ and $v(x,t)$, as unknowns variables still to be determined,

$$\frac{\partial p}{\partial x_i} + \frac{\partial v_i}{\partial t} + \sum_{j=1}^{3} v_j \frac{\partial v_i}{\partial x_j} = \nu\nabla^2 v_i \tag{83}$$

The Navier-Stokes equations without external force.

Equations (80) applied to (83) result in

$$\begin{cases} \frac{\partial p}{\partial x_i} + \frac{\partial v_i}{\partial t} + v_1\left(\frac{\partial v_1}{\partial x_1} + \alpha\frac{\partial v_1}{\partial x_2} + \beta\frac{\partial v_1}{\partial x_3}\right) = \nu\nabla^2 v_1 \\ \frac{\partial p}{\partial x_2} + \alpha\frac{\partial v_1}{\partial t} + \alpha v_1\left(\frac{\partial v_1}{\partial x_1} + \alpha\frac{\partial v_1}{\partial x_2} + \beta\frac{\partial v_1}{\partial x_3}\right) = \nu\alpha\nabla^2 v_1 \\ \frac{\partial p}{\partial x_3} + \beta\frac{\partial v_1}{\partial t} + \beta v_1\left(\frac{\partial v_1}{\partial x_1} + \alpha\frac{\partial v_1}{\partial x_2} + \beta\frac{\partial v_1}{\partial x_3}\right) = \nu\beta\nabla^2 v_1 \end{cases} \tag{84}$$

As

$$\begin{aligned} \frac{\partial v_1}{\partial x_1} + \alpha\frac{\partial v_1}{\partial x_2} + \beta\frac{\partial v_1}{\partial x_3} &= T(t)\frac{dX}{d\xi}\left(\frac{\partial \xi}{\partial x_1} + \alpha\frac{\partial \xi}{\partial x_2} + \beta\frac{\partial \xi}{\partial x_3}\right) \\ &= T(t)\frac{dX}{d\xi}(1+1-2) = 0 \end{aligned} \tag{85}$$

By the definition of we use in (80.1), then (84) becomes

$$\begin{cases} \frac{\partial p}{\partial x_i} + \frac{\partial v_i}{\partial t} = \nu\nabla^2 v_1 \\ \frac{\partial p}{\partial x_2} + \alpha\frac{\partial v_1}{\partial t} = \alpha\nu\nabla^2 v_1 \\ \frac{\partial p}{\partial x_3} + \beta\frac{\partial v_1}{\partial t} = \beta\nu\nabla^2 v_1 \end{cases} \tag{86}$$

or equivalently,

$$\begin{cases} \frac{\partial p}{\partial x_i} = \nu\nabla^2 v_1 - \frac{\partial v_1}{\partial t} \\ \frac{\partial p}{\partial x_2} = \alpha[\nu\nabla^2 v_1 - \frac{\partial v_1}{\partial t}] = \alpha\frac{\partial p}{\partial x_i} \\ \frac{\partial p}{\partial x_3} = \beta[\nu\nabla^2 v_1 - \frac{\partial v_1}{\partial t}] = \beta\frac{\partial p}{\partial x_i} \end{cases} \tag{87}$$

Similar to what we saw in section § 5, equation (24) for a=1 and b=0, we will make the pressure to be defined as

$$\frac{dp}{d\xi} = Q(\xi)R(t) \tag{88}$$

and the velocity

$$v_i = c_i X(\xi(x))T(t), c_1 = 1, c_2 = \alpha, c_3 = \beta \tag{89}$$

with defined in (80.1),

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$$\xi = x_1 + \frac{1}{\alpha}x_2 - 2\frac{1}{\beta}x_3, \alpha \neq 0, \beta \neq 0 \tag{90}$$

Then it will be sufficient, besides the equation (88) to the pressure, solve one linear partial differential equation involving v_1, instead of a system of three nonlinear partial differential equations involving v_1, v_2, v_3.

The development of the solution here follows the same steps already seen in section § 5, equations (29) to (43), being the main change the expression for ξ given in (90), with increased dimensions and the proportionality between v_2, v_3 and v_1. We come to

$$\begin{aligned} v(x,t) &= X(x_1 + \frac{1}{\alpha}x_2 - 2\frac{1}{\beta}x_3)T(t)(1,\alpha,\beta) \\ &= [A\cos(B\xi) + C\,sen(\pm B\xi)](1 - e^{-t})e^{-t}(1,\alpha,\beta), \alpha, \beta \neq 0 \end{aligned} \tag{91}$$

keeping valid the solutions (42) and (43) for the pressure p and velocity u, respectively. Initial velocity equal to(81). We also have the validity of $\nabla \cdot v=0$ and the corresponding integral $\int_{R^3} |v|^2\, dx$ infinite, portion of kinetic energy total system (1) to (6).

Conclusion

All three examples obey the necessary conditions of divergence-free ($\nabla \cdot u^0=0$), smoothness (C^∞) and partial derivatives of $u^0$and f of $C_{\alpha k}(1+|x|)^{-k}$ and $C_{\alpha m k}(1+|x|+t)^{-k}$ order, respectively. We conclude that we must have $u^0 \in S(R)^3$and $f \in S(R^3\times [0,))$. To each possible $u(x,t)$ so that (3) is true, the external force $f(x,t)$ and the pressure $p(x,t)$can be fittingly constructed, in C^∞class, verifying (8), and in a way to satisfy all the necessary conditions, finding, this way, a possible solution to (1), (2), (3), (4), (5) and (6), and only (7) wouldn't be satisfied for $t>0$, according to (10). We then show examples of breakdown solutions to case (C) of this millennium problem. These examples, however won't take to case A from [1] of existence and smoothness of solutions, because they violate (7) (case (A) also impose a null external force, $f=0$).

An overview of the problem's conditions is listed below (:R^3 and : $R^3\times$ [0,) representing the respective functions domains).

$v > 0$, $n=3$

$\exists u^0(x):\mathbb{R}^3$ smooth (C^∞), divergence-free ($\nabla\, u^0=0$)

$\exists f(x, t) : \mathbb{R}3\times[0, \infty)$ smooth (C^∞)

(4) $\left|\partial_x^{\alpha} u^0(x)\right| \le C_{\alpha k}\left(1+|x|\right)^{-k} : \mathbb{R}^3, \forall \alpha, K$

(5) $\left|\partial_x^{\alpha} \partial_t^{m} f(x,t)\right| \le C_{\alpha m K}\left(1+|x|+t\right)^{-k} : \mathbb{R}^3 \times [0,\infty), \forall \alpha, m, K$

$\nexists (p,u) : \mathbb{R}^3 \times [0,\infty)\,/$

(1) $\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} = v\nabla^2 u_i - \frac{\partial p}{\partial x_i} + f_i(x,t), 1 \le i \le 3\left(x \in \mathbb{R}^3, t \ge 0\right)$

(2) $\nabla\, u=0$

(3) $u\,(x,0)=u^0(x)$ $(x \in \mathbb{R}^3)$

(6) p, $u \in C^\infty(\mathbb{R}^3\times[0,\infty))$

(7) $\int_{\mathbb{R}^3} \left|u(x,t)\right|^2 dx < C, \forall t \ge 0$ (bounded energy)

In all three examples the head velocity u we used was of the form

$$u(x,t) = [u^0(x) + w(x)(1 - e^{-t})e^{-t} \tag{92}$$

in example 1, $w(x)=1$, in example 2, $w(x)=X(\xi)$ (1,1,0), $\xi=x_1-x_2$, and in example 3, $w(x)=X(\xi)$ $(1,\alpha,\beta)$, $\xi = x_1 \frac{1}{\alpha}x_2 - 2\frac{1}{\beta}x_3$, α,β cte. 0, examples 2 and 3 with $X(\xi)$ $[A\cos(B\xi) + C\,sen\,(\pm B\xi)]$.

It's important that we also analyse the solution's uniqueness question. As $u^0(x)$ and $f(x,t)$ are given of class C^∞, chosen by us, and satisfying (4) and (5), i.e., belonging to the Schwartz space, with $\cdot u^0=0$, claim that there is no solution (p,u) to the system (1), (2), (3), (6) and (7) might assume that we explored, or proved to, the infinite possible combinations of p and u, i.e., of (p,u). So we need that exists uniqueness of solution for the velocity that we build, eliminating other possible velocitys for the same data used, $u^0(x)$ and $f(x,t)$, and involving in finite total kinetic energy.

The uniqueness of the solution (except due the pressure $p\ (x,t)$ with constant additional term or time-dependent $\theta(t)$, além de outros casos de não unicidade da pressão sobre x e $T(t)$) comes from classical results already known, for example described in the mentioned article of Fefferman [1]: the system of Navier-Stokes equations (1), (2), (3) it has unique solution for all t 0or only for a finite time interval $[0,T)$ depending on the initial data, where T is called "blowup time". When there is a solution with finite T then the velocity u becomes unbounded near the "blowup time".

We see that the existence of each our solution in the given examples are guaranteed by construction and direct substitution. Our velocities has no irregular behavior, any regularity loss, at no time , in none position, that becomes one unlimited, infinite, even for t ∞ or $|x| \to \infty$, therefore, there can be no "blowup time" in the examples we gave, therefore the solutions found in the previous cases are unique at all times (unless pressure). But even if there were a finite T (in [14,15] we see that T > 0), the uniqueness would exist in at least a small interval of time, which is enough to show that in this time range occurs the breakdown of Navier-Stokes solutions because it was disobeyed the limited kinetic energy condition (7), making the case (C) true.

We must understand that uniqueness is in the main velocity u (equation 1), it is not necessary that is also in secondary velocity v (equations 14, 18 and 83), which as we have seen in the examples 2 and 3 it can have infinite solutions, due to the absence of n nonlinear terms $\sum_{j=1}^{n} v_j \frac{\partial v_i}{\partial v_j}$. Chosen a velocity v, however, applying in it the external force f (equations 13.4, 17.5, 82.4), results finally in the uniqueness of u(according 13.3, 17.4, 43, 82.3), solution of an equation with all the terms, of its kinetic energy and the corresponding divergence of the total kinetic energy $\int_{R^3}|u|^2dx$ in $t > 0$ due to the term $\int_{R^3}|v|^2dx \to \infty$. The pressure p, we already know, It is not unique, but this does not change, qualitatively, the fact that the total kinetic energy of the system is infinite or not. This is better understood with examples 1 and 2: v being any constant or time dependent exclusively, or with $x \in R$ or with $x \in R^2$, since not identically zero, and whatever the pressure , null or not, the condition (7) is violated due to integration of $|v|^2$ in the whole space R^3.

It is not difficult to extend the results obtained earlier in the § 5 with the two-dimensional speed to a speed v with three non-zero spatial components, as we saw in section § 8.

In examples 2 and 3 we had to solve an ordinary differential equation to get $X(\xi)$. We will now, however, find a solution non unique for the velocity in the Navier-Stokes equations, but without solving any differential equation aid. You just have to make an integration necessary to obtain pressure. Out of curiosity, the initial speed may be different from zero, as well as the external force, and we are not concerned to seek just endless kinetic energies or velocities belonging to the Schwartz space. We are not looking now a breakdown solution, on the contrary, we seek endless (many) solutions.

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We will solve the system (1), (2), (3) for the special case in which

$$\begin{aligned} u(x_1,x_2,x_3,t) &= X(x_1+x_2+x_3)T(t)(1,1,-2) = X(\xi)T(t)J \\ \xi(x) &= x_1+x_2+x_3, J=(1,1,-2) \end{aligned} \tag{93}$$

being worth $\nabla\cdot(\xi J)=0$. This gives us $\nabla\cdot u=\nabla\cdot u^0=0$ and the elimination of non-linear terms $(u\cdot\nabla)u \equiv \left(\sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j},\right)_{1\le i\le 3} = 0$ (of the Navier-Stokes equations, with or without external force. So the solution of (1) will be reduced to the solution of one linear partial differential equation, the Heat Equation three-dimensional inhomogeneous,

$$\frac{\partial p}{\partial x_i} = \nu\nabla^2 u_i - \frac{\partial u_i}{\partial t} + f_i = \phi_i, 1\le i\le 3 \tag{94}$$

need to be true

$$\frac{\partial \phi_i}{\partial x_j} = \frac{\partial \phi_j}{\partial x_i}, i\ne j \tag{95}$$

As $\frac{\partial p}{\partial x_i} = \frac{\partial p}{\partial \xi}\frac{\partial \xi}{\partial x_i} = \frac{\partial p}{\partial \xi}$, $\forall i$, as well the differential operators $\frac{\partial}{\partial x_i} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x_i} = \frac{\partial}{\partial \xi}$ and $\left(\frac{\partial}{\partial x_i}\right)^2 = \left(\frac{\partial}{\partial \xi}\right)^2$, $\forall i$, i.e., we have a pressure that may be expressed as a function of ξ, as well as the velocity components u, and the x_i are shown symmetrically and linearly relative to $\xi=x_1+x_2+x_3$, with the transformation of infinitesimal element of integration $d\xi = \frac{\partial \xi}{\partial x_1}dx_1 + \frac{\partial \xi}{\partial x_2}dx_2 + \frac{\partial \xi}{\partial x_3}dx_3 = dx_1+dx_2+dx_3$, equality (95) is true, it is valid $\frac{\partial^2 p}{\partial x_j \partial x_i} = \frac{\partial^2 p}{\partial x_i \partial x_j}$, and we have the following solution to the pressure:

$$p(x,t) - p_0(t) = \int_{i_0}^{\xi(x)} \left(\nu T\nabla^2 X - X\frac{dT}{dt} + f\right) d\xi \tag{96}$$

with

$$\frac{\partial p}{\partial x_1} = \frac{\partial p}{\partial x_2} = \frac{\partial p}{\partial x_3} \tag{97}$$

assuming that the force $f(x,t)$ is of the form $Y(\xi)Z(t)$ (1,1,-2), such as $u(x,t)=X(\xi)T(t)$ (1,1,-2). Let us consider $p_0(t)$ as the pressure at the instant t and the surface $\xi=\xi_0$. This solves the system we wanted, since the integration in (96) is possible, and so we do not solve any intermediate ordinary differential equation to find $X(\xi)$, because we can prefix which the expression for $X(\xi)$ we want to use, among infinite possibilities, and such that have $u(x,0)=u^0(x)$.

Other combinations of the components of the vector J may be used, as well as other combinations of the coefficients of x_1,x_2,x_3 in ξ, provided that they eliminate non-linear terms and check it (2) and (95). Thus, more complex forms to ξ are also possible, in addition to linear, which provides a robust way to achieve solutions for u. For example, defining

$$u_i = \alpha_i(x,t)u_1, 1\le i\le n, \alpha_1 = 1 \tag{98}$$

the condition to be obeyed by X and ξ in order to eliminate the nonlinear terms is

$$\alpha_i \frac{dX(\xi)}{d\xi}\sum_{j=1}^{n}\alpha_j\frac{\partial \xi}{\partial x_j} + X(\xi)\sum_{j=1}^{n}\alpha_j\frac{\partial \alpha_i}{\partial x_j} = 0 \tag{99}$$

for all (natural) in $1\le i\le n$. For each determined i liminates the nonlinear term of the line (or coordinate) i if (99) is satisfied.

One way to do (99) true is when

$$\sum_{j=1}^{n}\alpha_j\frac{\partial \xi}{\partial x_j} = \sum_{j=1}^{n}\alpha_j\frac{\partial \alpha_i}{\partial x_j} = 0 \tag{100}$$

When the α_j are constant or time dependent only the condition to be obeyed for ξ is

$$\sum_{j=1}^{n}\alpha_j\frac{\partial \xi}{\partial x_j} = 0 \tag{101}$$

which is in accordance with examples 2 and 3 above.

Including also the incompressibility condition for u, must be valid also the relation

$$\begin{aligned} \sum_{j=1}^{n}\frac{\partial(\alpha_j u_1)}{\partial x_j} &= u_1\sum_{j=1}^{n}\frac{\partial \alpha_j}{\partial x_j} + \sum_{j=1}^{n}\alpha_j\frac{\partial u_1}{\partial x_j} \\ &= T(t)\left[X(\xi)\sum_{j=1}^{n}\frac{\partial \alpha_j}{\partial x_j} + \frac{dX(\xi)}{d\xi}\sum_{j=1}^{n}\alpha_j\frac{\partial \xi}{\partial x_j}\right] = 0 \end{aligned} \tag{102}$$

As (102) must be valid for all t, then we need to be valid

$$X(\xi)\sum_{j=1}^{n}\frac{\partial \alpha_j}{\partial x_j} + \frac{dX(\xi)}{d\xi}\sum_{j=1}^{n}\alpha_j\frac{\partial \xi}{\partial x_j} = 0 \tag{103}$$

When the α_j are constant or time dependent only, the condition to be obeyed for ξ is equal to the condition (101) previous,

$$\sum_{j=1}^{n}\alpha_j\frac{\partial \xi}{\partial x_j} = 0 \tag{104}$$

Notice that the function $T(t)$ in (93) must not have singularities in case it is desired that the velocity u is regular, limited in module, notwithstanding, $T(t)$ singular, infinite for one or more values of time , the function can be considered as a "highlighter" of blowups, and so we can build solutions with instants of blowup τ_* well determined, to our will, such that $T(\tau_*) \to \infty$.

In the absence of singularities of $T(t)$ and $X(\xi(x))$, however, only wishing regular velocities, it follows that it is possible for a three-dimensional Navier-Stokes equation (generally, n-dimensional) "well behaved" have more than one solution for the same initial velocity. To the special form given to the solution $u(x,t)$ in (93), with T(0)=0 or not, for a same initial velocity $u(x,0)=X(\xi(x))T(0)J=u^0(x)$with $J=(1,1,-2)$, can be generated, in principle, infinite different velocities $u(x,t)=X(\xi(x))T(t)J$, for different functions of the position $X(\xi(x))$ and $T(t)$ time , solutions that solve the Navier-Stokes equation (1). If the external force is zero, this brings us to the negative answer to 15th problem of Smale [12], as we have seen before thinking only in the non uniqueness of pressure due to the additional term $\theta(t) + q$, where $q \ne 0$is a constant and $\theta(t)$ an explicit function of time (in the original Smile problem pressure does not vary in time).

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