

Verification of Some Properties of the C-nilpotent Multiplier in Lie Algebras

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Abstract

The purpose of this paper is to obtain some inequalities and certain bounds for the dimension of the c-nilpotent multiplier of finite dimensional nilpotent Lie algebras and their factor Lie algebras. Also, we give an inequality for the dimension of the c-nilpotent multiplier of L connected with dimension of the Lie algebras $\gamma_d(L)$ and $L/Z_{d-1}(L)$. Finally, we compare our results with the previously known result.

Keywords: C-nilpotent multiplier; Nilpotent lie algebra; Lie algebra

Introduction

All Lie algebras referred to in this article are (of finite or infinite dimension) over a fixed field F and the square brackets $[,]$ denotes the Lie product. Let $0 \rightarrow R \rightarrow F \rightarrow L \rightarrow 0$ be a free presentation of a Lie algebra L, where F is a free Lie algebra. Then we define the, c-nilpotent multiplier $c \geq 1$, to be

$$M^{(c)}(L) = \frac{(R \cap \gamma_{c+1}(F))}{\gamma_{c+1}(R, F)},$$

where $\gamma_{c+1}(F)$ is the (c+1)-th term of the lower central series of F, $\gamma_1(R, F) = 1$ and $\gamma_{c+1}(R, F) = [\gamma_c(R, F), F]$. This is analogous to the definition of the Baer-invariant of a group with respect to the variety of nilpotent groups of class at most c given by Baer [1-3] (for more information on the Baer invariant of groups). The Lie algebra $M^{(1)}(L) = (R \cap F^2) / [R, F] = M(L)$ is the most studied Schur multiplier of L [4,5]. It is readily verified that the Lie algebra $M^{(c)}(L)$ is abelian and independent of the choice of the free presentation of L [6]. The purpose of this paper is to obtain some inequalities for the dimension of the c-nilpotent multiplier of finite dimensional nilpotent Lie algebras and their factor Lie algebras (Corollary 2.3 and Corollary 2.5). Finally, we compare our results to upper bound given [6]. First, we show that for each ideal N in L, there is a close relationship between the $M^{(c)}(L)$ and $M^{(c)}(L/N)$

Lemma 1.1. Let L be Lie algebra with a free presentation $0 \rightarrow R \rightarrow F \rightarrow L \rightarrow 0$. If S is an ideal in F with $N \cong S/R$, then the following sequences are exact:

$$(i) \quad 0 \rightarrow \frac{R \cap \gamma_{c+1}(S, F)}{\gamma_{c+1}(R, F)} \rightarrow M^{(c)}(L) \rightarrow M^{(c)}(L/N) \rightarrow \frac{N \cap \gamma_{c+1}(L)}{\gamma_{c+1}(N, L)} \rightarrow 0,$$

$$(ii) \quad 0 \rightarrow (N)_{ab} \otimes^c (L/L^2)_{ab} \rightarrow M^{(c)}(L) \rightarrow M^{(c)}(L/N) \rightarrow N \cap \gamma_{c+1}(L) \rightarrow 0.$$

under the condition that N is central, $N \otimes^c M = N \otimes \underbrace{M \otimes \dots \otimes M}_{c\text{-times}}$ and $(N)_{ab} = N/N^2$.

Proof. We prove only part (ii). Since N is central, $[S, F] \subseteq R$ and

$$\gamma_{c+1}(S, F^2 + R) \subseteq \gamma_{c+1}(S, R) + \gamma_{c+1}(R, F^2) \subseteq \gamma_{c+1}(R, F) + \gamma_{c+1}(R, [F, F]) \subseteq \gamma_{c+1}(R, F).$$

Now, we have the following homomorphism

$$\alpha : S/R \otimes^c F/(F^2 + R) \rightarrow \frac{R \cap \gamma_{c+1}(F)}{\gamma_{c+1}(R, F)}$$

such that $\text{Im } \alpha = \gamma_{c+1}(S, F) / \gamma_{c+1}(R, F)$. Now the result holds by part (i).

The following corollary is an immediate consequence of Lemma 1.1, which gives some elementary results about dimension of the c-nilpotent multiplier of finite dimensional Lie algebras see corollary 2.2 of Salemkar et al. [6].

Corollary 1.2. Let N be an ideal of Lie algebra L. Then

$$(i) \quad \dim M^{(c)}(L) + \dim(N \cap \gamma_{c+1}(L)) = \dim M^{(c)}(L/N) + \dim \left(\frac{\gamma_{c+1}(S, F)}{\gamma_{c+1}(R, F)} \right),$$

Where F, S, R are defined in Lemma 1.1.

$$(ii) \quad \dim M^{(c)}(L) + \dim(N \cap \gamma_{c+1}(L)) \leq \dim M^{(c)}(L/N) + \dim \left((N)_{ab} \otimes^c (L/L^2)_{ab} \right)$$

Suppose that L is generated by n elements. Let F be a free Lie algebra generated by n elements and $L \cong F/R$ Witt's formula from Bahturin et al. [7] gives us

$$\dim \gamma_d(F) / \gamma_{d+1}(F) = \frac{1}{d} \sum_{m|d} \mu(m) n^m = l_n(d),$$

where $\mu(m)$ is the Mobius function, defined by $\mu(1) = 1$, $\mu(k) = 0$ if K is divisible by a square, and $\mu(p_1 \dots p_s) = (-1)^s$ if p_1, \dots, p_s are distinct prime numbers.

Lemma 1.3. Let L be an abelian Lie algebra of dimension n. Then $\dim M^{(c)}(L) = l_n(c+1)$. In particular, $\dim M(L) = \frac{1}{2}n(n-1)$.

Proof. Consider a free Lie algebra F freely generated by n elements. By Witt's formula, F/F^2 is an abelian Lie algebra of dimension n, and so it is isomorphic to L. Hence $\dim M^{(c)}(L) = \dim(\gamma_{c+1}(F) / \gamma_{c+2}(F))$, which gives the result.

Let $L = \gamma_1(L) \supseteq \gamma_2(L) \supseteq \dots \supseteq \gamma_c(L) \supseteq \gamma_{c+1}(L) = 0$ be the lower central series of nilpotent Lie algebra, L. L is said to have class c if c is the least integer for which $\gamma_{c+1}(L) = 0$. Furthermore, if $\dim \gamma_j(L) / \gamma_{j+1}(L) = 1$ for $j=2, 3, \dots, c$ and $\dim L / \gamma_2(L) = 2$, then L is said to be of maximal class c. Additionally, let $0 = Z_0(L) \subset Z_1(L) \subset Z_2(L) \subset \dots \subset Z_c(L) = L$ be the upper central series of nilpotent Lie algebra L. If L is of maximal

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class, then $Z_i(L) = \gamma_{c-i+1}(L)$ for $0 \leq i \leq c$.

By the above notation we have the following corollary.

Corollary 1.4. Let L be a finite dimensional nilpotent Lie algebra of maximal class $(c+1)$, then

$$\dim M^{(c)}(L) \leq \dim M^{(c)}(L/Z(L)) + 2^c - 1$$

Proof. Using Corollary 1.2(ii) with $N = Z(L)$, we get

$$\dim M^{(c)}(L) + \dim Z(L) = \dim M^{(c)}(L) + 1 \leq \dim M^{(c)}(L/Z(L)) + (1)2^c.$$

Discussion and Results

2 Bounds on $\dim M^{(c)}(L)$

Let L be a finite dimensional nilpotent Lie algebra of class $d \geq 2$. First, we give an inequality for the dimension of the c -nilpotent multiplier of L connected with dimension of the Lie algebras $\gamma_d(L)$ and $L/Z_{d-1}(L)$ (Corollary 2.3) and some inequalities for the dimension of the c -nilpotent multiplier of finite dimensional nilpotent Lie algebras will be given. For this purpose, we need the following two lemmas.

Lemma 2.1. Let H and N be ideals of Lie algebra L and $N = N_0 \supseteq N_1 \supseteq \dots$, a chain of ideals of N such that $[N_i, L] \subseteq N_{i+1}$ for all $i = 1, 2, \dots$. Then

$$[N_i, [H, L]] \subseteq N_{i+j+1} \text{ for all } i, j.$$

Proof. We have

$$\begin{aligned} [N_i, [H, L]] &\subseteq [N_i, [[H, L], L]] \\ &\subseteq [[N_i, [H, L]], L] + [[N_i, L], [H, L]] \\ &\subseteq [N_{i+j+1}, L] + [N_{i+1}, [H, L]] \\ &\subseteq N_{i+j+2} + N_{i+j+2} = N_{i+j+2}. \end{aligned}$$

Now, the assertion follows by induction on j .

Lemma 2.2. Let L be a finite dimensional nilpotent Lie algebra of class $d \geq 2$. Let $0 \rightarrow R \rightarrow F \rightarrow L \rightarrow 0$ be a free presentation of L , then $\frac{\gamma_{c+1}(\gamma_d(F) + R, F)}{\gamma_{c+1}(R, F)}$ is a homomorphic image of $\gamma_d(L) \otimes \frac{L}{Z_{d-1}(L)} \otimes \dots \otimes \frac{L}{Z_{d-1}(L)}$.

Proof. Put $Z_k(L) = T_k/R$ for $0 \leq k \leq d$. Now consider the following chain

$$S = T_d \supseteq \dots \supseteq T_k \supseteq T_{k-1} \supseteq \dots \supseteq T_1 \supseteq T_0 = R.$$

Since $[T_k, F] \subseteq T_{k-1}$, then by Lemma 2.1,

$$[T_{d-1}, [\gamma_{d-2}(F), F]] \subseteq T_{d-1-(d-2+1)} = T_0 = R.$$

Therefore,

$$\begin{aligned} [\gamma_d(F) + R, \underbrace{T_{d-1}, \dots, T_{d-1}}_{c\text{-times}}] &\subseteq [\gamma_d(F), \underbrace{T_{d-1}, \dots, T_{d-1}}_{c\text{-times}}] + [R, \underbrace{T_{d-1}, \dots, T_{d-1}}_{c\text{-times}}] \\ &\subseteq [\gamma_d(F), \underbrace{T_{d-1}, \dots, T_{d-1}}_{c\text{-times}}] + \gamma_{c+1}(R, F) \\ &\subseteq [[[T_{d-1}, F], \gamma_{d-1}(F)], \underbrace{T_{d-1}, \dots, T_{d-1}}_{(c-1)\text{-times}}] \\ &\quad + [[[T_{d-1}, [\gamma_{d-2}(F), F]], F], \underbrace{T_{d-1}, \dots, T_{d-1}}_{(c-1)\text{-times}}] + \gamma_{c+1}(R, F) \end{aligned}$$

$$\subseteq [[[T_{d-2}, \gamma_{d-1}(F)], \underbrace{T_{d-1}, \dots, T_{d-1}}_{(c-1)\text{-times}}] + \gamma_{c+1}(R, F)$$

\vdots

$$\subseteq [[[T_0, \gamma_1(F)], \underbrace{T_{d-1}, \dots, T_{d-1}}_{(c-1)\text{-times}}] + \gamma_{c+1}(R, F)$$

$$\subseteq \gamma_{c+1}(R, F).$$

The latter inclusion gives the following epimorphism

$$\frac{\gamma_d(F) + R}{R} \times \underbrace{\frac{F}{T_{d-1}} \times \dots \times \frac{F}{T_{d-1}}}_{c\text{-times}} \rightarrow \frac{\gamma_{c+1}(\gamma_d(F) + R, F)}{\gamma_{c+1}(R, F)}$$

$$(x + R, f_1 + T_{d-1}, \dots, f_c + T_{d-1}) \mapsto [x, f_1, \dots, f_c] + \gamma_{c+1}(R, F).$$

Corollary 2.3. Under the assumptions and notation of the above Lemma, we have

$$\dim M^{(c)}(L) + \dim (\gamma_d(L) + \gamma_{c+1}(L)) \leq \dim M^{(c)}(L/\gamma_d(L)) + \dim \gamma_d(L) \left[\dim \left(\frac{L}{Z_{d-1}(L)} \right) \right]^c.$$

Proof. In Corollary 1.2(i), taking $N = \gamma_d(L) = \frac{\gamma_d(F) + R}{R}$. Now by Lemma 2.2, we have

$$\begin{aligned} \dim M^{(c)}(L) + \dim (\gamma_d(L) + \gamma_{c+1}(L)) &= \dim M^{(c)}(L/\gamma_d(L)) + \dim \left(\frac{\gamma_{c+1}(\gamma_d(F) + R, F)}{\gamma_{c+1}(R, F)} \right) \\ &\leq \dim \left(\gamma_d(L) \otimes \underbrace{\frac{L}{Z_{d-1}(L)} \otimes \dots \otimes \frac{L}{Z_{d-1}(L)}}_{c\text{-times}} \right) \\ &\quad + \dim M^{(c)}(L/\gamma_d(L)) \\ &= \dim M^{(c)}(L/\gamma_d(L)) + \dim \gamma_d(L) \left[\dim \left(\frac{L}{Z_{d-1}(L)} \right) \right]^c. \end{aligned}$$

In following, we give another an inequality for the dimension of the c -nilpotent multiplier of finite dimensional nilpotent Lie algebras.

Theorem 2.4. Let L be a finite dimensional nilpotent Lie algebra of class ≥ 1 , then

$$\dim M^{(c)}(L) \leq \dim M^{(c)}(L/L^2) + \dim L^2 [\dim(L/Z(L)) - \dim(L/Z(L))^2]^c - \dim(\gamma_{c+1}(L)).$$

Proof. We use induction on the class of L . If L is of class 1, then $L^2 = 0$ and the result holds. Assume the result for nilpotent Lie algebras of class to be less than d and let L have class $m = d-1$. Note that $\gamma_m(L) \subseteq Z(L)$, $L^2 \subseteq Z_{m-1}(L)$, $(L/\gamma_m(L))^2 = L^2/\gamma_m(L)$ and $Z(L)/\gamma_m(L) \subseteq Z(L/\gamma_m(L))$. For convenience, let $A = (L/\gamma_m(L))/Z(L/\gamma_m(L))$ and $B = L/Z(L) = (L/\gamma_m(L))/(Z(L)/\gamma_m(L))$. Since A is a homomorphic image of B , it follows that $\dim A/A^2 \leq \dim B/B^2$. By induction,

$$\begin{aligned} \dim M^{(c)}(L/\gamma_m(L)) &\leq \dim M^{(c)}\left(\frac{L/\gamma_m(L)}{(L/\gamma_m(L))/Z(L/\gamma_m(L))}\right) \\ &\quad + \dim(L/\gamma_m(L))^2 \left[\dim(A/A^2) \right]^c - \dim(\gamma_{c+1}(L/\gamma_m(L))) \\ &\leq \dim M^{(c)}(L/L^2) + \dim(L^2/\gamma_m(L)) \left[\dim(B/B^2) \right]^c - \dim(\gamma_{c+1}(L/\gamma_m(L))). \end{aligned}$$

By Corollary 2.3,

$$\dim M^{(c)}(L) \leq \dim M^{(c)}(L/\gamma_m(L)) + \dim(\gamma_m(L)) \left[\dim(L/Z_{m-1}(L)) \right]^c - \dim(\gamma_m(L) \cap \gamma_{c+1}(L)).$$

$$\text{Also, } \dim(L/Z_{m-1}(L)) \leq \dim(L/(L^2 + Z(L))) = \dim(B/B^2).$$

Therefore,

$$\begin{aligned} \dim M^{(c)}(L) &\leq \dim M^{(c)}(L/L^2) + \dim(L^2/\gamma_m(L)) \left[\dim(B/B^2) \right]^c - \dim(\gamma_{c+1}(L/\gamma_m(L))) \\ &\quad + \dim(\gamma_m(L)) \left[\dim(B/B^2) \right]^c - \dim(\gamma_m(L) \cap \gamma_{c+1}(L)) \\ &\leq \dim M^{(c)}(L/L^2) + \dim L^2 \left[\dim(B/B^2) \right]^c - \dim(\gamma_{c+1}(L)). \end{aligned}$$

Since $\dim(B/B^2) \leq \dim(L/L^2)$, we obtain:

Corollary 2.5. Under the assumptions and notation of the above Theorem, we have

$$\dim M^{(c)}(L) \leq \dim M^{(c)}(L/L^2) + \dim L^2 \left[\dim(L/L^2) \right]^c - \dim \gamma_{c+1}(L).$$

Now, we compare our results to upper bound given [7], when $c = 1$.

Theorem 2.6. Let L be a finite dimensional nilpotent Lie algebra of class m and $d=d(m)$. Then

$$\dim M(L) \leq \sum_{k=1}^m I_d(k+1).$$

Example 2.7. Let F be a free Lie algebra on 2 generators and $L = F/F^3$. Then L is a Lie algebra of 2 generators and class 2. Thus $\dim L/L^2 = I_2(1) = 2, \dim L^2/L^3 = I_2(2) = 1$ and $\dim L = 3$. By Theorem 2.6,

$$\dim M(L) \leq \sum_{j=1}^2 I_2(j+1) = I_2(2) + I_2(3) = 1 + \frac{1}{3}(\mu(1)2^3 + \mu(3)2) = 1 + \frac{1}{3}(6) = 3.$$

Note that L is a finite dimensional nilpotent Lie algebra of maximal class $(1 + 1)$ and $Z(L) = L^2$. By Corollary 1.4 and Lemma 1.3,

$$\dim M(L) \leq \dim M(L/Z(L)) + 1 = \dim M(L/L^2) + 1 = I_2(2) + 1 = 1 + 1 = 2.$$

Also, by Corollary 2.5,

$$\dim M(L) \leq \dim M(L/L^2) + \dim L^2 [\dim(L/L^2)] - \dim L^2 = I_2(2) + 1(2) - 1 = 2.$$

Example 2.8. Let F be a free Lie algebra on 2 generators and

$L = F/F^4$. Then L is a Lie algebra of 2 generators and class 3. Thus $\dim L = 5, \dim L^2 = 3$ and $\dim L/L^2 = I_2(1) = 2$. By Theorem 2.6

$$\begin{aligned} \dim M(L) &\leq \sum_{j=1}^3 I_2(j+1) = I_2(2) + I_2(3) + I_2(4) \\ &= 1 + 2 + \frac{1}{4}(\mu(1)2^4 + \mu(2)2^2 + \mu(4)2) \\ &= 1 + \frac{1}{4}(16 - 4) = 6. \end{aligned}$$

Also, by Corollary 2.5,

$$\dim M(L) \leq \dim M(L/L^2) + \dim L^2 [\dim(L/L^2)] - \dim L^2 = I_2(2) + 3(2) - 3 = 4.$$

In this two examples, we see that our results give two better upper bounds for $\dim(L)$ than the previously known result.

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