The Automorphism Groups and Derivation Algebras of Two-Dimensional Algebras

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Abstract

The automorphism groups and derivation algebras of all two-dimensional algebras over algebraically closed fields are described.

Keywords: Algebra; Isomorphism; Structure constants; Automorphism; Derivation

MSC(2010): Primary: 17A36; 17B40; Secondary: 14R20; 14L30

Introduction

There is a long history and an extended study of the automorphism groups of algebras. Determining the full automorphism group of an algebra is generally an immensely difficult problem. For instance, the automorphism group of the polynomial ring of three variables is not yet understood, and a result in this direction is given by Shestakov and Umirbaev [1]. Since 1990, many researchers have successfully computed the automorphism groups of interesting infinite-dimensional noncommutative algebras, including certain quantum groups, generalized quantum Weyl algebras, skew polynomial rings and many more; the results are given [2-7], which is only a partial list. Recently, by using a rigidity theorem for quantum tori, Yakimov has proved the Andruskiewitsch-Dumas conjecture and the Launois-Lenagan conjecture [8,9]. A uniform approach to both the Andruskiewitsch-Dumas conjecture and the Launois-Lenagan conjecture is given in a preprint by Goodearl and Yakimov [10]. Authors [11] use the discriminant to determine the automorphism groups of some noncommutative algebras. Note that most of the results above are obtained for algebras over the fields of real or complex numbers. The study of the automorphism groups and derivation algebras is of a big interest due to the importance of them in the study of structure of algebras. This is one of the motivations to describe the automorphism groups and derivation algebras of all 2-dimensional algebras over any algebraically closed field.

Authors [12] have presented a complete list of isomorphism classes of two-dimensional algebras over algebraically closed fields, providing a list of canonical representatives of their structure constant’s matrices. In the present paper we describe the groups of automorphisms and derivation algebras of all those listed algebras.

In fact, the automorphism groups of all 2-dimensional algebras have been given earlier [13]. In contrast to that we provide an explicit realization of the automorphism groups for the all listed canonical algebras [12].

The first part of this paper (Sections 2 and 3) is devoted to the description of the groups of automorphisms and the second part (Section 4) deals with the derivation algebras. In each case we consider problems over algebraically closed fields of characteristic not 2,3, characteristic 2 and characteristic 3 separately according to the classification results given [12].

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\( \mathbf{u} \cdot \mathbf{v} = e(A(\mathbf{u} \otimes \mathbf{v})) \)

for \( u = (u_1, u_2, \ldots, u_n) \), \( v = (v_1, v_2, \ldots, v_n) \) and \( w = (w_1, w_2, \ldots, w_n) \) are column coordinate vectors of \( \mathbf{u} \), \( \mathbf{v} \), respectively. The matrix \( A \in M(m \times m; \mathbb{F}) \)

defined above is called the matrix of structural constants (MSC) of \( \mathbb{A} \) with respect to the basis \( \mathbf{e} \). Further we assume that a basis \( \mathbf{e} \) is fixed and we do not make a difference between the algebra \( \mathbb{A} \) and its MSC \( A \).

An automorphism \( \mathbb{A} \rightarrow \mathbb{A} \) as an invertible linear map is represented by an invertible matrix \( A \in GL(m; \mathbb{F}) \); \( g(\mathbf{u}) = A(\mathbf{u}) = \mathbf{eu} \). Due to:

\[
\mathbf{g}(\mathbf{u} \cdot \mathbf{v}) = (gA)(\mathbf{u} \otimes \mathbf{v}) = g(A(\mathbf{u} \otimes \mathbf{v})),
\]

and

\[
\mathbf{g}(\mathbf{u}) \cdot \mathbf{g}(\mathbf{v}) = (gA)(\mathbf{u} \otimes \mathbf{v}) = e(A^2)(\mathbf{u} \otimes \mathbf{v})
\]

the property \( \mathbf{g}(\mathbf{u} \mathbf{v}) = \mathbf{g}(\mathbf{u}) \mathbf{g}(\mathbf{v}) \) is equivalent to:

\[
gA = Ag^{2}. \tag{1}
\]

An derivation \( \mathbb{d} : \mathbb{A} \rightarrow \mathbb{A} \) as a linear map is represented by a matrix \( d \in M(m; \mathbb{F}) \) as follows \( \mathbf{d}(\mathbf{u}) = d(\mathbf{u}) = ed(\mathbf{u}) \).

Due to:

\[
d(\mathbf{u} \cdot \mathbf{v}) = e(A(\mathbf{u} \otimes \mathbf{v}))) = ed(A(\mathbf{u} \otimes \mathbf{v}))) = e(dA)(\mathbf{u} \otimes \mathbf{v}) = e(dA)(\mathbf{u}) \otimes e(dA)(\mathbf{v})
\]

and

\[
d(\mathbf{u}) \cdot \mathbf{v} + \mathbf{u} \cdot d(\mathbf{v}) = (ed\mathbf{u})(\mathbf{v}) + eA(\mathbf{u} \otimes d\mathbf{v}) = eA(\mathbf{u} \otimes (d\mathbf{v})) + ed\mathbf{u}(\mathbf{v})
\]

the property \( \mathbf{d}(\mathbf{u} \cdot \mathbf{v}) = \mathbf{d}(\mathbf{u}) \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{d}(\mathbf{v}) \) is equivalent to:

\[
dA = A(dI + I \otimes dA), \tag{2}
\]

where \( I \) stands for the identity matrix.

If \( \mathbf{e}' = (e'_{1}, e'_{2}, \ldots, e'_{m}) \) is another basis of \( \mathbb{A} \), \( \mathbf{e}' \mathbf{g} = \mathbf{e} \) with \( g \in G = GL(m; \mathbb{F}) \), and \( \mathbb{A}' \) is MSC of \( \mathbb{A} \) with respect to \( \mathbf{e}' \) then it is known that:

\[
A'g = A(g^{-1}) = \tag{3}
\]

is valid. Thus, the isomorphism of algebras \( \mathbb{A} \) and \( \mathbb{B} \) over \( \mathbb{F} \) given above can be rewritten as follows.

**Definition 2.5**

Two \( m \)-dimensional algebras \( \mathbb{A} \) and \( \mathbb{B} \) over \( \mathbb{F} \), given by their matrices of structure constants \( A \) and \( B \), are said to be isomorphic if \( B = gA(g^{-1}) \) holds true for some \( g \in GL(m; \mathbb{F}) \).

Further we consider only the case \( m = 2 \) and for the simplicity we use:

\[
A = \begin{pmatrix}
\alpha_{1} & \alpha_{2} & \alpha_{3} & \alpha_{4} \\
\beta_{1} & \beta_{2} & \beta_{3} & \beta_{4}
\end{pmatrix}
\]

for MSC, where \( \alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}, \beta_{1}, \beta_{2}, \beta_{3}, \beta_{4} \) stand for any elements of \( \mathbb{F} \).

Due to [1] we have the following classification theorems according to \( \text{Char}(\mathbb{F}) = 2, 3 \), \( \text{Char}(\mathbb{F}) = 2 \) and \( \text{Char}(\mathbb{F}) = 3 \) cases, respectively.

**Theorem 2.6**

Over an algebraically closed field \( \mathbb{F} \) (\( \text{Char}(\mathbb{F}) = 2 \) and \( 3 \)), any non-trivial 2-dimensional algebra is isomorphic to only one of the following algebras listed by their matrices of structure constants:

\[
A_{2}(e) = \begin{pmatrix}
\alpha_{1} & \alpha_{2} & \alpha_{3} & \alpha_{4} \\
\beta_{1} & \beta_{2} & \beta_{3} & \beta_{4}
\end{pmatrix}
\]

for MSC, where \( \alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}, \beta_{1}, \beta_{2}, \beta_{3}, \beta_{4} \) stand for any elements of \( \mathbb{F} \).

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\[
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\alpha_{1} & \alpha_{2} & \alpha_{3} & \alpha_{4} \\
\beta_{1} & \beta_{2} & \beta_{3} & \beta_{4}
\end{pmatrix}
\]

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The Groups of Automorphisms of 2-Dimensional Algebras

Due to eqn. (1) for the group of automorphisms of an algebra A given by MSC A∈M(2×4; F) one has:

\[ \text{Aut}(A) = \{ g \in GL(2; F) : gA = A \} \quad \text{if} \quad g \neq 1. \]

Therefore further we look only for nonsingular solutions of the eqns. (3) and (6) imply
\[ a^2 + b^2 + c^2 = 0, \quad \text{and} \quad d^2 + e^2 + f^2 = 0. \]

Due to the first two equalities one has
\[ a^2 + b^2 = 0, \quad \text{and} \quad d^2 + e^2 = 0. \]

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Note that the proof of eqn. (10) does not depend on characteristics of $F$.

Let $A = A_0(\alpha, \beta, \beta_1, \beta_2) = \begin{pmatrix} \alpha_0 & 0 & 1 \\ \beta & 1 - \alpha & 0 \end{pmatrix}$.

Due to eqn. (4) one has the system of equations:
\[-c^2 + a\alpha - a^2\alpha + b\beta_1 = 0,\]
\[-cd - ab\alpha + b\beta_2 = 0,\]
\[-b - cd - ab\alpha - ab\alpha = 0,\]
\[-a - d^2 - b\alpha = 0,\]
\[-ac + c\alpha + ac\alpha - a^2\beta + d\beta_1 - ac\beta_2 = 0,\]
\[-bc + bca_1 - ab\beta + d\beta_1 - ad\beta = 0,\]
\[d - ad - da\alpha + ad\alpha = 0,\]
\[c - bd - bda\alpha - \beta_1 - ad\beta = 0.\]

From the eqns. (2) and (3) of the system (11) we get $b(1 - \beta_1) = 0$. The following cases occur:

**Case 1:** $1 - \alpha, \beta_1 = 0$. Then $b = 0$ and one gets $c = 0$, so $a = 1 - a = 0$ and $d(1 - a)(1 - a) = 0$ due to the eqns. (1) and (7), respectively. Thus $a = 1$. The eqn. (4) implies $d = 1$ and $\beta_1(d - a^2) = 0$ due to the eqn. (5). We get two cases:

Case 1.1. $\beta_1 = 0$. Then $d = 1$, therefore $g$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Case 1.2. $\beta_1 \neq 0$. Then $d = 1$, so $g = I$.

**Case 2:** $\beta_2 = 1$, $\alpha_1 = 0$. If $b = 0$ it is easy to see that $g$ equals

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

If $\beta_1 \neq 0$. Now one can assume that $b = 0$, so due to the eqns. (4) and (1) we have

$$\alpha_1 = \frac{a - d^2}{b}, \quad \beta_1 = \frac{c^2 + b^2 + a^2 + d^2 - a - d^2}{b^2},$$

respectively. The substitution them into eqn. (11). The results of the following system of equations:

\[-a - a^2 + b^2 - bcd + d^2 + a^2d^2 = 0,\]
\[a^2 - a + abc + 2abbc - 2abcd - abcd - a - d + a^2 - d^2 = 0,\]
\[+b^2c^2 - d^2 + a^2d^2 + b^2d^2 - 2abcd + ad^2 + a^2d^2 = 0,\]
\[-a^2 - a + ab - b - a^2b - b^2c^2 + a^2c + b^2d^2 - abcd + d^2 + a^2d^2 = 0,\]
\[-a^2 - a + bc + b - a^2c + 2ad + b - a^2d^2 - 2abcd + ad^2 + a^2d^2 = 0,\]
\[-a^2 - a + bc + b - a^2c + 2ad + b - a^2d^2 - 2abcd + ad^2 + a^2d^2 = 0.\]

Now we make the use of the eqns. (2) and (4) to get:

$$abc + a^2bc + 2abbc - 2abcd + a^2d^2 + b^2d^2 - bcd^2 - 2abcd + ad^2 + a^2d^2 = 0,$$

that is:

$$h(a + a^2 + bcd - d^2) = 0.$$

Then the eqn. (1) of the system gives:

$$b^2c^2 = a + d^2 - 2abd^2 - 2abcd + a^2d^2 = 0.$$

This implies $a = \frac{1}{2}$. The substitution it into the system (12) yields:

$$\begin{align*}
3a^2 + 4bc + 6bc^2 - 8bc^2 - 8bc^2 & = 0, \\
1 + 32bc - 32bc^2 - 8bc^2 - 4d & = 0, \\
1 + 8bc - 16bc^2 - 8bc^2 & = 0, \\
1 + 8bc & = 0.
\end{align*}$$

Due to the eqns. (1) and (2) of the system of eqn. (13) one gets:

$$1 + 9bc - 8bc^2 - 4d - 2d^2 - 16bc^2 + 8d^2 = 0$$

and therefore taking into account the eqn. (3), one has $8b^2c^2 - 4bd - 4b(2bc - d^2) = 0$, i.e., $d = 2bc$. This implies $\Delta = ad - bc - bc - bc = 0$, i.e., $g$ is singular. Therefore:

$$\text{Aut}(A_0(\alpha, \beta, \beta_1, \beta_2)) = \{1\}, \text{ if } \beta_1 \neq 0$$

and

$$\text{Aut}(A_0(\alpha_1, \beta_1, \beta_2)) = \{1, 0\}.$$
where

\[ \alpha \neq 0 \]

and

\[ b = 0 \]

\[ c = 0 \]

\[ d \neq 0 \]

It is easy to see that for this system \( b = 0 \) (so \( a \neq 0, d \neq 0 \) due to \( \Delta \neq 0 \)), i.e.,

\[ a\alpha (1-\alpha) = 0, \]

\[ b\beta (1-\beta) = 0, \]

\[ d \neq 0 \]

hence \( a = 1 \). Therefore, \( c = 0 \) and \( d = 0 \) by the eqn. (5).

**Case 1:** \( \beta = 2\alpha - 1 \). We get \( g = \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \) where \( d \neq 0 \).

**Case 2:** \( \beta \neq 2\alpha - 1 \). In this case \( c = 0 \) and we obtain \( g = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \), where \( d \neq 0 \).

Let \( A = A_1(\alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 1 & 2\alpha_1 - 1 & 1 - \alpha_1 \end{pmatrix} \) 

From eqn. (4) we have the system of equations:

\[ b + a\alpha, - a\alpha^2, \]

\[ -b + 2a\alpha - a\alpha^2, \]

\[ b - b\alpha, - a\alpha^2, \]

\[ -b\alpha^2, \]

\[ -2a^2 + b + c\alpha, - a\alpha^2, \]

\[ -a^2 + d + c\alpha, - a\alpha^2, \]

\[ -a^2 + d + c\alpha, - a\alpha^2, \]

\[ -2a^2 + b + c\alpha, - a\alpha^2, \]

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Here immediately we get \( b = 0, a = 1, d = 1 \) and \( g = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \).

Let \( A_1(\alpha_1, \beta_1) = \begin{pmatrix} \alpha_1 & 0 & 0 \\ \beta_1 & 1 & 1 - \alpha_1 \end{pmatrix} \).

Then eqn. (4) gives the system of equations:

\[ -c^2 + a\alpha, - a^2 + b\beta, \]

\[ b - cd - b\alpha, - a\alpha^2, \]

\[ -cd - b\alpha, - a\alpha^2, \]

\[ a - d^2 - b\alpha^2, \]

\[ -ac + c\alpha, 2a\alpha^2, - a^2 + b\beta, \]

\[ a^2 + c\alpha, 2a\alpha^2, - a^2 + b\beta, \]

\[ b + d - 2a\alpha^2, - a\alpha^2, - ab\beta, \]

\[ -bc + c\alpha, - d\alpha, + ab\beta, \]

\[ c + b\alpha, - d\alpha, + ab\beta, \]

\[ c + b\alpha, - d\alpha, + ab\beta, \]

It is easy to see from the system that \( c = 0, b = 0 \) and \( a = 1 \). The eqns. (4) and (5), imply \( d^2 = 1, \beta_1(d - 1) = 0 \), therefore we have to consider the following two cases:

**Case 1:** \( \beta_1 = 0 \). We get \( d = \pm 1 \) and \( g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

**Case 2:** \( \beta_1 = 0 \). We obtain \( d = 1 \) and \( g = I \).

Let \( A_1(\beta_1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \). Then:

\[ gA_1(\alpha_1, \beta_1) = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 & 0 & 0 \\ \beta_1 & 1 & 1 - \alpha_1 \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 1 & 2\alpha_1 - 1 & 1 - \alpha_1 \end{pmatrix} \]

According to eqn. (4) one gets \( b = 0, d = 1 \), that is \( a = 1 \), \( c + 3a\alpha = 0 \). Again we consider two cases:

**Case 1:** \( \alpha_1 = 1 \). We get \( g = \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \) where \( d \neq 0 \).

**Case 2:** \( \alpha_1 \neq 1 \). Then \( c = 0 \) and \( g = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \), where \( d \neq 0 \).

Let \( A_1(\alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 1 & 1 - \alpha_1 & 0 \end{pmatrix} \).

Then:

\[ gA_1(\alpha_1, \beta_1) = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 & 0 & 0 \\ \beta_1 & 1 & 1 - \alpha_1 \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 1 & 2\alpha_1 - 1 & 1 - \alpha_1 \end{pmatrix} \]

Owing to eqn. (4) one has \( b = 0, a = 1, d = 1 \) and \( g = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \).

Let \( A_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \). Then:

\[ gA_1 = A_1(g \otimes g) = \begin{pmatrix} a_1 & d & c+cd \end{pmatrix} \cdot \begin{pmatrix} a_1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \]

Due to eqn. (4) it is easy to see that \( c = 0, d = 1 \) and \( b = 0 \) i.e.,

\[ g = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \]

where \( a \neq 0 \).

Let \( A_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \). Then:

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The eqn. (4) gives \( b(1+2d) = 0 \). The following cases may occur:

**Case 1:** \( b = 0 \). Then one has \( c = 0, d = 1, a = \pm 1 \) and \( g = \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix} \).

**Case 2:** \( d = -\frac{1}{2} \). In this case \( b = \pm \frac{\sqrt{3}}{2}, a = \pm \frac{1}{2} \) and \( c = \frac{b}{2a} \).
Case 2.1. $a = \frac{1}{2}$ Then $c = \pm \frac{\sqrt{3}}{2}$ and $g = \left( \begin{array}{cc} 1 & \pm \frac{\sqrt{3}}{2} \\ \pm \frac{\sqrt{3}}{2} & -1 \end{array} \right)$.

Case 2.2. $a = -\frac{1}{2}$ Then $c = \pm \frac{\sqrt{3}}{2}$ and $g = \left( \begin{array}{cc} 1 & \pm \frac{\sqrt{3}}{2} \\ \pm \frac{\sqrt{3}}{2} & 1 \end{array} \right)$.

Let $A = A_2 = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{array} \right)$ Then:

$gA_2 - A_2 (g \otimes g) = \left( \begin{array}{ccc} b & 0 & 0 \\ -a^2 + d & -ab - ab & -b^2 \end{array} \right)$

Due to eqn. (4) one has $g = \left( \begin{array}{c} 1 \\ c \end{array} \right)$, where $a \neq 0$.

Here are the corresponding results in the cases of characteristic 2 and 3. The proof is similar to that of the case of characteristic not 2 and 3.

**Theorem 3.2**

The automorphism groups of the algebras listed in Theorem 2.7 are given as follows:

$\text{Aut}(A_2(\alpha, \beta)) = \{ I \}$,
$\text{Aut}(A_2(\beta, \beta)) = \{ I \}$,
$\text{Aut}(A_2(\alpha, \beta)) = \left\{ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right\}$,
$\text{Aut}(A_2(\alpha, \beta)) = \left\{ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right\}$.

**Remark 3.4**

Another interpretation of Theorem 4.1 (Theorem 4.2, Theorem 4.3) is the description of stabilizers of the matrices listed in Theorem 2.6 (respectively, Theorem 2.7, Theorem 2.8), with respect to the action (3).

**Derivations of 2-dimensional Algebras**

If $A$ is an algebra given by MSC $A$ then, due to eqn. (2) the algebra of its derivations $\text{Der}(A)$ is represented as follows:

$\text{Der}(A) = \{ D \in M(2, F) : A(D \otimes I + I \otimes D) - DA = 0 \}$. (16)

Further we use the notation $D = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$.

One of the main results of this section is given in the following theorem.

**Theorem 4.1**

The derivations of all algebra structures on 2-dimensional vector space over an algebraically closed field $F$ of characteristic not 2.3 are given as follows:

$\text{Der}(A(\alpha, \beta, \gamma)) = \text{Der}(A(\gamma, \beta, \alpha)) = \text{Der}(A(\beta, \alpha, \gamma)) = \{ 0 \}$,
$\text{Der}(A(\alpha, \gamma)) = \left\{ \begin{array}{cc} 0 & 0 \\ 0 & d \end{array} \right\}$ if $\beta + 2\gamma - 1$,
$\text{Der}(A(\alpha, \beta, \gamma)) = \left\{ \begin{array}{cc} 0 & d \\ 0 & d \end{array} \right\}$ if $\beta + 2\gamma - 1$,
Proof. Let \( A = A_4(\alpha_1, \alpha_2, \alpha_3, \beta_1) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 + 1 & \alpha_4 \\ \beta_1 & -\alpha_1 & -\alpha_3 + 1 & -\alpha_2 \end{pmatrix} \).

Then the equality \( A_4(\alpha_1, \alpha_2, \alpha_3, \beta_1) (D \otimes 1 + 1 \otimes D) - D A_4(\alpha_1, \alpha_2, \alpha_3, \beta_1) = 0 \) is equivalent to the following system of equations:

\[
\begin{align*}
\alpha_1 + \alpha_2 + 2c\alpha_2 - b\beta_1 &= 0, \\
c - 3c\alpha_1 + 2a\beta_1 - d\beta_1 &= 0, \\
2c\beta_1 + d\alpha_3 + c\alpha_4 &= 0, \\
-\alpha_1 - 2c\alpha_2 + b\beta_1 &= 0, \\
-c + 2b\alpha_1 - d\beta_1 &= 0, \\
-a + 3a\alpha_3 - 2d\alpha_4 &= 0, \\
3b &= 0, \\
b - 2b\alpha_1 + b\beta_1 &= 0.
\end{align*}
\]

The eqns. (3) and (8) of the system of equations above imply \( b = 0 \), the eqns. (1) and (6) imply \( a = 0 \), the eqns. (3) and (5) imply \( d = 0 \) and the eqns. (1) and (4) imply \( c = 0 \), therefore we get \( D = 0 \).

Let \( A = A_4(\alpha_1, \alpha_2, \alpha_3, \beta_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & \beta_2 & 1 - \alpha_1 & 0 \end{pmatrix} \).

Then the eqn. (16) is equivalent to the system of equations:

\[
\begin{align*}
\alpha_1 - b\beta_1 &= 0, \\
c - 2c\alpha_1 + 2a\beta_1 - d\beta_1 + c\beta_2 &= 0, \\
c + b\alpha_1 - b\beta_2 &= 0, \\
2b\beta_1 + a\beta_2 &= 0, \\
-a - 3a\alpha_3 + 2d\alpha_4 &= 0, \\
-b + 2a\alpha_1 - d\beta_1 &= 0, \\
-b + 2b\alpha_1 - a\alpha_3 &= 0, \\
\alpha_1 + b\beta_1 &= 0.
\end{align*}
\]

The eqns. (3) and (7) imply \( b = 0 \), the eqns. (1) and (6) give \( a = 0 \), then according to the eqn. (2) we have \( d = 0 \). Therefore, \( D = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \).

Let \( A_4(\alpha_1, \alpha_2, \beta_1, \beta_2) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & \beta_2 & 1 - \alpha_1 & 0 \end{pmatrix} \).

Then due to eqn. (16) one has the system of equations:

\[
\begin{align*}
2c - b\beta_1 &= 0, \\
2a\beta_1 - d\beta_1 + c\beta_2 &= 0, \\
d - b\beta_2 &= 0, \\
-2c + b\beta_1 + a\beta_2 &= 0, \\
-a + 2d &= 0, \\
3b &= 0, \\
-b + d + b\beta_2 &= 0.
\end{align*}
\]

The eqns. (5) and (7) imply \( b = d = 0 \), the eqn. (1) gives \( c = 0 \), the eqn. (6) implies \( a = 0 \), hence \( D = 0 \).

Let \( A = A_4(\alpha_1, \alpha_2, \beta_1, \beta_2) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \beta_1 & 1 - \alpha_1 & 0 \end{pmatrix} \).

Then the eqn. (16) is equivalent to the system of equations:

\[
\begin{align*}
\alpha_1 &= 0, \\
c - 2c\alpha_1 + b\beta_1 &= 0, \\
a\beta_1 &= 0, \\
2a\beta_1 - d\beta_1 &= 0, \\
-a + 2d &= 0, \\
-b + d &= 0.
\end{align*}
\]
Let $A=A_{1}(a_{i})=egin{pmatrix} a_{1} & 0 & 0 \\ 0 & -a_{2} & 0 \\ 0 & -a_{3} & 0 \end{pmatrix}$. Then:

$$A_{1}(a_{i})(D \otimes I + I \otimes D) - DA_{1} = \begin{pmatrix} a_{1} & -b & 2a_{1} \\ -c & a_{2} - 2a_{1} & 0 \\ 0 & 2a_{3} & 0 \end{pmatrix}.$$
dimensional algebras over algebraically closed fields are described through the all theorems and proofs.

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References