Unconjugated Contact Forms

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Abstract

In this paper, we prove the existence of infinitely many number non-isomorphic contact structures on the torus $T^3$. Moreover, these structures are explicitly given by $\omega_n = \cos n\theta_1 \, d\theta_1 + \sin n\theta_1 \, d\theta_2 \, (n \in \mathbb{N})$.

Keywords: Contact structures; Reeb field; Poisson brackets

Introduction

In the acts of Colloquium of Brussels in 1958, Libermann [1] addressed the study of the automorphisms of the contact structures on a differentiable manifold $M$. She has proved that these automorphisms correspond bijectively to functions on this manifold. This allows to transport the Lie algebra structure on the vector space $\mathbb{F}(M)$ of the functions on $M$. We obtain, for given functions $f, g \in \mathbb{F}(M)$, a Poisson bracket $\{f, g\}$ that depends of the contact form $\omega$. The study of the infinite dimensional Lie algebras obtained is far from be advanced. Thus, in 1973 Lichnerowicz [2] who hopes distinguish the contact structures by their Lie algebras, has given series of results that are all however of general character. Some works that have appeared have emphasis on the similarities of these algebras. In 1979, Lutz [3] has proved the existence of infinitely many non-isomorphic contact structures on the sphere $S^n$. In 1989, as reported by Lutz [3] himself, we have opened in our thesis [4] new perspectives in the other direction by studying the sub-algebras of finite dimension of these algebras. We know that if two contact structures $[\omega_1]$ and $[\omega_2]$ are isomorphic then their Lie algebras (of infinite dimension of course) $\mathcal{A}(\omega_1)$ and $\mathcal{A}(\omega_2)$ are also isomorphic.

Given an $n$-dimensional smooth manifold $M$, and a point $p \in M$, a contact element of $M$ with contact point $p$ is an $(n-1)$-dimensional linear subspace of the tangent space to $M$ at $p$: A contact contact element can be given by the zeros of a $1$-form on the tangent space to $M$ at $p$: However, if a contact element is given by the zeros of a $1$-form $\omega$, then it will also be given by the zeros of $\lambda \omega$ where $\lambda \neq 0$: thus $\{\lambda \omega: \lambda \neq 0\}$ gives the same contact element. It follows that the space of all contact elements can be given by the zeros of a $1$-form on the tangent space to $M$ at $p$: However, if a contact element is given by the zeros of a $1$-form $\omega$, then it will also be given by the zeros of $\lambda \omega$ where $\lambda \neq 0$: thus $\{\lambda \omega: \lambda \neq 0\}$ gives the same contact element.

For more details, we can consult the previous studies [5-8].

The Main Result

The main result is contained in the following theorem:

**Theorem 1**

On the torus $T^3$ the contact structures defined by the contact forms $\omega_n = \cos n\theta_1 \, d\theta_1 + \sin n\theta_1 \, d\theta_2 \, (n \in \mathbb{N})$ are non-isomorphic.

To establish this result, we need the following lemma.

**Lemma 2**

Let $f$ a $C^\infty$-function on the torus $T^3$ and $R_\ast$ the Reeb field of $\omega_n$ defined by $R_\ast = \cos \theta_1 \, \partial / \partial \theta_1 + \sin \theta_1 \, \partial / \partial \theta_2$.

If $R_\ast$ is zero, then $f$ depend only on $\theta_2$.

**Proof:** $R_\ast$ is zero means that $f$ is constant along the integral curves of $R_\ast$ whose equations are:

$$
\frac{\partial \theta_1}{\partial t} = \cos \theta_1,
\frac{\partial \theta_2}{\partial t} = \sin \theta_1,
\frac{\partial \theta_3}{\partial t} = 0.
$$

So, we have,

$$
\theta_1 = t \cos k_1 + k_1,
\theta_2 = t \sin k_1 + k_1,
\theta_3 = k_1,
$$

where $k_1$, $k_2$, and $k_3$ are real constants.

When $\tan k_1$ is irrational, the trajectories are dense on a torus $T^3$; so by continuity $f$ is constant on this torus. Hence, we get $\frac{\partial f}{\partial \theta_1} = \frac{\partial f}{\partial \theta_2} = 0$.

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for \( \theta_1, \theta_2 \) arbitrary and \( \theta_3 \) in a dense subset of the circle. It follows that \( f \) is constant with respect to \( \theta_1 \) and \( \theta_2 \).

This completes the proof of the lemma.

**Proof of the theorem:** It suffices to prove that the structures \([\omega_n]\) and \([\omega_n']\) are non-isomorphic.

From a study [1] we recall that the Poisson brackets associated to \([\omega_n]\) and \([\omega_n']\) are given respectively by:

\[
\{ f, g \} = \int \left( \frac{\partial f}{\partial \theta_i} \frac{\partial g}{\partial \theta_j} - \frac{\partial f}{\partial \theta_j} \frac{\partial g}{\partial \theta_i} \right) \cos \theta_i \sin \theta_j d\theta_i d\theta_j \]

for \( i, j = 1, 2 \), and consequently the function \( w \) depend only on \( \theta_3 \).

Define the function \( w(\theta_1, \theta_2, \theta_3) \) as follows from \( T^3 \) into \( T^3 \) by:

\[
[w(\theta_1, \theta_2, \theta_3)]_{i=1,2} = \int \frac{\partial f}{\partial \theta_i} g d\theta_i + \int \frac{\partial g}{\partial \theta_i} f d\theta_i \sin \theta_i \cos \theta_j d\theta_i d\theta_j
\]

Thus we have \([\Phi, \Psi]_1 = \Omega, [\Psi, \Omega]_1 = \Phi \) and \([\Omega, \Phi]_1 = -\Psi \).

If \( \Phi(\theta_1, \theta_2, \theta_3) = \sin \theta_3, \Psi(\theta_1, \theta_2, \theta_3) = \cos \theta_2 \) and \( \Omega(\theta_1, \theta_2, \theta_3) = -\sin \theta_1 \) then \( \Phi, \Psi \) and \( \Omega \) generate a three dimensional sub-algebra of \( \Omega \) isomorphic to \( \text{SL}_2(\mathbb{C}) \).

Let \( \Phi, \Psi \) and \( \Omega \) generate a three dimensional sub-algebra of \( \Omega \) isomorphic to \( \text{SL}_2(\mathbb{C}) \).

Thus, we have by analogy

\[
\omega = \frac{\partial}{\partial \theta_i} \cos \theta_i + \frac{\partial}{\partial \theta_j} \sin \theta_j = \lambda \cos 2\theta_i, \quad \text{where} \quad \lambda = \mathbb{C}
\]

This completes the proof of the theorem.

**Conclusion**

The techniques used in this work to find nonisomorphic contact structures can be extended to the sphere \( S^3 \) in a first step and may be to other manifolds suitably chosen. It is also interesting to find the group of diffeomorphisms that leaves the contact structure invariant.

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**References**