An Extended Jacobian Elliptic Function Expansion Approach to the Generalized Fifth Order KdV Equation

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Abstract

In the present study, the well-known extended Jacobian elliptic function expansion approach is successfully employed to reveal the exact travelling wave solutions of the generalized fifth order KdV (GKdV) equation. The travelling wave solutions are derived in forms rational, hyperbolic and exponential functions. The achieved results are correctly confirmed by presenting some 2D and 3D figures. The proposed technique is concise and effective, and simply applicable mathematical concept for constructing the exact solutions of most nonlinear wave equations emerged in some natural phenomena.

Keywords: Extended jacobian elliptic function expansion method; Generalized fifth order KdV equation; Solitary wave solutions


Introduction

A massive number of dynamical problems appeared in mathematical physics, engineering, biology, nuclear physics, mechanics, optics and chemistry can be often modelled by using nonlinear partial differential equations (NPDEs) [1-8]. Thus, it is necessary to construct the exact or numerical solutions of NPDEs. In recent decades, there has been an utterly immense interest in establishing theoretical techniques by which one can find the travelling wave solutions. On other words, obtaining the exact solutions of NPDEs has become one of the broadest scientific subjects. Some specialists discovered some methods such as the Jacobi elliptic function method [9,10], tanh-sech method [11,12], homogeneous balance method [13,14], exp-function method [15,16], Hirota’s bilinear transformation scheme [17], Riccati-Bernoulli sub-ODE method [18,19], sine-cosine method [20-22], F-expansion method [23,24], extended tanh-method [25], G/G′-expansion method [26,27] and many others. It should be observed that the mentioned methods are not applicable for some NPDEs.

In this paper, we aim to use the Jacobi elliptic function method to understand the travelling wave solutions for the following GKdV equation [28,29] from a theoretical standpoint:

\[ G_t + \mu_1 G G_x + \mu_2 G = \frac{\mu_2}{3} (G^3_x) + G_{xxx} = 0. \]

(1)

Where \( \mu_1, \mu_2 \) and \( \mu_3 \) are arbitrary constants which their values will modify the GKdV equation characteristics. Eq. (1) is investigated by some experts. For instance, the authors in a study [30] utilized the inverse scattering method to present the solution of Eq. (1) when \( \mu_2=10, \mu_3=-20 \) and \( \mu_4=30 \) Seadawy et al. [31] expressed the solutions and stability of Eq. (1) for \( \mu_2=15, \mu_2=45 \) and \( \mu_3 \) was assumed to be constant. In particular, \( \mu_3 \) was taken by -15 and 75/2. Various cases of solutions were presented according to the values of \( \mu_3 \). When \( \mu_3=\mu_2=\mu_2 \), then Eq. (1) is formed as

\[ G_t + \mu_2 (G G_x) + \frac{\mu_2}{3} (G^3_x) + G_{xxx} = 0. \]

(2)

For example, the standard SK equation is given by Eq. (2) when \( \mu_2=\mu_2=5 \). It was solved by using the exp-function method [32]. Ali [33] employed the generalized exp(-\( \phi(\zeta) \)) expansion method to extract the exact wave solutions of the standard SK equation. In a study [34], the invariant, symmetry and exact solutions of Eq. (2) for \( \mu_2=15 \) and \( \mu_2=45 \) are developed by using Lie symmetry analysis method.

The extended Jacobian elliptic function expansion method [9,10] is described and utilized to extract the exact wave solutions of Eq. (1) and Eq. (2). It is the critical purpose of this analysis to show the validity and accuracy of the proposed technique in obtaining the solutions. The considered method played a prominent role in nonlinear evolution equations is a reliable, accurate, powerful and effective mathematical tool for finding the exact solutions of several NPDEs.

The outline of this article is given as follows. In Section 2, the proposed approach is deeply described and suggested some possible solutions. Section 3 is mainly devoted to analyse the GKdV equation and extract some relevant exact solutions. The travelling wave solutions are presented in various cases. Finally, Section 4 concludes this work and shows the essential results.

Analysis of the Extended Jacobian Elliptic Function Expansion Method

An extensive description of the extended Jacobian elliptic function expansion approach [9,10] is illustrated in this section. We present the proposed mathematical concept in some steps listed as follows.

- Consider a given NPDEs in \( x,t \) on the form

\[ L(p, p_x, p_t, p_{xx}, \ldots) = 0, \]

and seek a solution on the form

\[ p = p(\zeta), \quad \zeta = k(x-ct). \]

(4)

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Here, $k$ and $c$ represent the wave number and wave speed, respectively.

- Using Eq. (4), Eq. (3) is simply converted to the following ODE:
  \[ Q(p, p_c, p_{cc}, p_{ccc}, \ldots) = 0, \]
  \[ (5) \]
  Where $Q$ is a polynomial in $p(\zeta)$, and
  \[ p_c = \frac{dp}{d\zeta}, \quad p_{cc} = \frac{d^2p}{d\zeta^2}, \]
  and so on.

- The considered scheme presents the solutions on the form
  \[ p(\zeta) = a_0 + \sum_{j=1}^{4} f_j(\zeta) + h_j g_j(\zeta), \]
  \[ j=1,2,3,\ldots, \]
  with
  \[ f_j(\zeta) = \sinh^2(\zeta), \quad g_j(\zeta) = \cosh(\zeta), \]
  \[ f_j(\zeta) = \sinh(\zeta), \quad g_j(\zeta) = \cosh(\zeta). \]
  \[ (7) \]
  \[ \zeta \quad \zeta \quad \zeta \]
  \[ \zeta \quad \zeta \quad \zeta \]

where the Jacobi elliptic sine, cosine and tan functions are denoted by $\sinh(\zeta)$, $\cosh(\zeta)$, and $\tanh(\zeta)$ respectively. The relationships of these functions are provided as follows:

- The value of $N$ is evaluated by balancing the highest order linear term with nonlinear term as illustrated in the equations
  \[ r_0 N + r_1 (N + k). \]
  \[ (13) \]
  \[ \zeta \quad \zeta \quad \zeta \]
  \[ \zeta \quad \zeta \quad \zeta \]

It has been confirmed that the extended Jacobian elliptic function expansion technique is more dependable and appropriate than the mentioned schemes in a study [35].

**Results and Discussion**

In this section, the exact travelling wave solutions of GkDV equation are discussed and obtained. Eq. (1) can be easily written as

\[ G + \mu_1 (G G_{\zeta}) + \delta (G^2) + \frac{\mu_2}{3} (G^3) + G_{\zeta\zeta\zeta} = 0, \]

where $\delta = \frac{\mu_0 - \mu_1}{2}$. The wave transformation

\[ G(x,t) = G(\zeta), \quad \zeta = x - w t, \]

is used with Eq. (18), where $w$ is the speed of the wave, to arise the following ODE:

\[ -w G + \mu_1 (G G_{\zeta}) + \delta (G^2) + \frac{\mu_2}{3} (G^3) + G_{\zeta\zeta\zeta} = 0. \]

We now balance the highest order derivative $G_{\zeta\zeta\zeta}$ and non-linear term $G^3$ to end up with $m = 2$. Since $m = 2$, the proposed technique introduces the exact solutions on the form

\[ G(\zeta) = a_0 + a_1 \sinh(\zeta) + b_1 \cosh(\zeta) + a_2 \sinh^2(\zeta) + b_2 \cosh^2(\zeta), \]

Where $a_0$, $a_1$, $b_1$, $a_2$, $b_2$ are some constants shown later. From (21) we have

\[ G(\zeta) = \frac{a_1 \sinh(\zeta) + 2a_2 \cosh(\zeta) - b_1 \sinh(\zeta) - b_2 \cosh(\zeta)}{a_2 \sinh(\zeta) + b_2 \cosh(\zeta)}, \]

\[ (22) \]

\[ G_{\zeta\zeta\zeta} = -8a_0 - 8a_1 \sinh(\zeta) + 4b_1 \cosh(\zeta) + a_1 \sinh(\zeta) + 14a_2 \cosh(\zeta) + a_2 \sinh(\zeta) + 16b_2 \cosh^2(\zeta), \]

\[ (23) \]

\[ G_{\zeta\zeta\zeta\zeta} = -8a_0 - 8a_1 \sinh(\zeta) + 4b_1 \cosh(\zeta) + a_1 \sinh(\zeta) + 14a_2 \cosh(\zeta) + a_2 \sinh(\zeta) + 16b_2 \cosh^2(\zeta), \]

\[ + 44b_1 \cosh^4(\zeta) + 4b_2 \cosh(\zeta) + 16a_2 \cosh^2(\zeta), \]

\[ (24) \]

\[ G_{\zeta\zeta\zeta\zeta\zeta} = -8a_0 - 8a_1 \sinh(\zeta) + 4b_1 \cosh(\zeta) + a_1 \sinh(\zeta) + 14a_2 \cosh(\zeta) + a_2 \sinh(\zeta) + 16b_2 \cosh^2(\zeta), \]

\[ - 20a_2 \cosh^2(\zeta) - 8a_2 \cosh^2(\zeta) - 20a_2 \cosh^2(\zeta), \]

\[ - 120b_2 \cosh^4(\zeta) - 60b_2 \cosh^4(\zeta) - 120b_2 \cosh^4(\zeta) + 24b_2 \cosh^4(\zeta) + 120b_2 \cosh^4(\zeta) + 5b_2 \cosh^4(\zeta), \]

\[ (25) \]

\[ G_{\zeta\zeta\zeta\zeta\zeta\zeta} = -8a_0 - 8a_1 \sinh(\zeta) + 4b_1 \cosh(\zeta) + a_1 \sinh(\zeta) + 14a_2 \cosh(\zeta) + a_2 \sinh(\zeta) + 16b_2 \cosh^2(\zeta), \]

\[ + 44b_1 \cosh^4(\zeta) + 4b_2 \cosh(\zeta) + 16a_2 \cosh^2(\zeta), \]

\[ (26) \]

\[ G_{\zeta\zeta\zeta\zeta\zeta\zeta\zeta} = -8a_0 - 8a_1 \sinh(\zeta) + 4b_1 \cosh(\zeta) + a_1 \sinh(\zeta) + 14a_2 \cosh(\zeta) + a_2 \sinh(\zeta) + 16b_2 \cosh^2(\zeta), \]

\[ - 20a_2 \cosh^2(\zeta) - 8a_2 \cosh^2(\zeta) - 20a_2 \cosh^2(\zeta), \]

\[ - 120b_2 \cosh^4(\zeta) - 60b_2 \cosh^4(\zeta) - 120b_2 \cosh^4(\zeta) + 24b_2 \cosh^4(\zeta) + 120b_2 \cosh^4(\zeta) + 5b_2 \cosh^4(\zeta), \]

\[ (27) \]

\[ \mu_1 = \mu_2, \]
Second case In this case, we determined the constants as follows:

\[ a_1 = \frac{\sqrt{2} + \sqrt{11}}{\sqrt{2}}, \quad a_2 = b_2 = 0, \quad a_4 = -\frac{8(\sqrt{2} \pm \sqrt{11})}{\sqrt{2} \sqrt{2} + \sqrt{11}}, \quad \mu_1 = \sqrt{\mu_2}. \]  

The second family of equation is given by

\[ G(\zeta) = a_1 + a_2 \text{sn}^2(\zeta). \]  

Thus, the solution of Eq. (29) is expressed by

\[ G(x,t) = \frac{\sqrt{2} \pm \sqrt{11}}{\sqrt{2}} \text{sn}(x-wt)^2 - \frac{8(\sqrt{2} \pm \sqrt{11})}{\sqrt{2} \sqrt{2} + \sqrt{11}}, \quad \mu_1 = \sqrt{\mu_2}. \]  

Third case Here, the constants are solved as follows:

\[ a_1 = \frac{\sqrt{2} + \sqrt{11}}{\sqrt{2}}, \quad a_2 = b_2 = 0, \quad a_4 = -\frac{8(\sqrt{2} \pm \sqrt{11})}{\sqrt{2} \sqrt{2} + \sqrt{11}}, \quad \mu_1 = \sqrt{\mu_2}. \]  

Then, the third family of equation is

\[ G(\zeta) = a_1 + a_2 \text{sn}^2(\zeta) + b_2 \text{sn}(\zeta) \text{cn}(\zeta). \]  

Hence, Eq. (32) has the following solution:

\[ G(x,t) = \frac{\sqrt{2} \pm \sqrt{11}}{\sqrt{2}} \text{sn}(x-wt)^2 + \frac{8(\sqrt{2} \pm \sqrt{11})}{\sqrt{2} \sqrt{2} + \sqrt{11}}, \quad \mu_1 = \pm \sqrt{\mu_2}. \]

Figure 2 shows 2D and 3D plots for the solution of Eq. (30) when \( x_0=10, \mu_2=45 \).

Figure 1 shows 2D and 3D graphs for the solution of Eq (27) when \( x_0=20, \mu_2=45, w=16 \).

Figure 2: 2D and 3D plots for the solution of Eq. (30) when \( x_0=10, \mu_2=45 \).

Figure 3: The plot of the real part of Eq. (33) when \( x_0=20, \mu_2=45 \).
Figure 4 shows the plot of the imaginary part of Eq. (33) under the values $x_0=20, \mu_2=45$.

Conclusion

This article has been written to apply the extended Jacobian elliptic function expansion technique on the GKhV equation. The exact travelling wave solutions of Eq. (18) has been surely determined. Some complex solutions are also given as can be seen in above figure. The 2D and 3D figures emphasize the validity of the method. The proposed approach, which can be applied to other NPDEs, gives valuable, advantageous and successful results.

References


