Examples of Freudenthal-Kantor triple systems

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Abstract
Symmetry group of Lie algebras and superalgebras constructed from (ε, δ) Freudenthal-Kantor triple systems has been studied. Also, the definition and examples of hermitian triple systems is introduced in this note.

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Introduction

It seems that the concept of a triple system (or called a ternary algebra) in nonassociative algebras started from the metasymplectic geometry due to Freudenthal. After a generalization of the concept has been studied by Tits, Koecher, Kantor, Yamaguti, Allison and authors ([1,2] for many earlier references on the subject). Also it is well known the object of investigation of Jordan and Lie algebras with application to symmetric spaces or domains [3] and to physics [4,5].

Nonassociative algebras are rich in of mathematics, not only for pure algebra differential geometry, but also for representation theory and algebraic geometry. Specialy, the Lie algebras and Jordan algebras plays an important role in many mathematical and physical objects. As a construction of Lie algebras as well as Jordan algebras, we are interested in characterizing the Lie algebras from view point of triple systems [6-9]). These imply that we are considering to structure of the subspace of the five graded Lie (super)algebra \( L (c, \delta) \) satisfying \( L_1 \oplus L_2 \oplus L_3 \) associated with an (c,\( \delta \)) Freudenthal-Kantor triple system which contains a class of Jordan triple systems related 3 graded Lie algebra \( L_1 \oplus L_2 \oplus L_3 \). For these considerations without utilizing properties of root systems or Cartan matrices, we would like to refer to the articles of the present author and earlier references quoted therein [1,2,10,11].

In particular, for an characterizing of Lie algebras, an applying to geometry and physics, we will introduce couple topics about a symmetry of Lie algebras (section 1) and a definition of hermitian triple systems (section 2) in this note, which is a survey and an announcement of new results.

More precisely speaking, first, the symmetry group of Lie algebras and superalgebras constructed from (c,\( \delta \)) Freudenthal-Kantor triple systems has been studied. Especially, for a special (c, \( \delta \)) Freudenthal-Kantor triple, it is \( SL(2) \) group. Secondly, we will give a definition of hermitian generalized Jordan triple systems and the examples of their tripotent defined by elements \( C \) of triple systems satisfying \( C^3 = C \).

Symmetry of Lie algebras associated with triple systems

A triple system \( V \) is a vector space over a field \( \mathbb{F} \) of characteristic \( \neq 2 \) or 3 with a trilinear map \( \mathbb{F}^3 \rightarrow \mathbb{F} \). We denote the trilinear product ( or ternary product) by juxtaposition \( [x,y,z] \in \mathbb{F}^3 \).

A well studied triple system is the (c,\( \delta \)) Freudenthal-Kantor triple systems ( abbreviated hereafter as to (c,\( \delta \)) FKTS) with \( c \) and \( \delta \) being either +1 or -1 [1,2,9,12].

Since (c,\( \delta \)) Freudenthal-Kantor triple systems offer a simple method of constructing Lie algebras (for the case of (\( \delta \)=+1)) and Lie superalgebras (for the case of \( \delta =-1 \)) with 5-graded structure, it may be of some interest to study its symmetry group in this note. In order to facilitate the discussion, let us briefly sketch its definition.

We introduce two linear mappings \( L : V \otimes V \rightarrow V \) and \( K : V \otimes V \rightarrow V \) by

\[
L(x,y) = x)y, K(x,y) = - \delta yzx (1.1)
\]

for \( \delta = +1 \text{ or } -1 \) If they satisfy

\[
[L(u,v), L(x,y)] = L(L(u,v)x,y) + \epsilon L(x,L(u,v)y), (1.2)
\]

\[
K(K(u,v)x,y) = L(y,x)K(u,v)- \epsilon K(u,v)L(x,y) (1.3)
\]

for any \( u,v,x,y \in V \) and \( \epsilon = +1 \text{ or } -1 \) we call the triple system to be (c,\( \delta \)) FKTS.

One consequence of Eqs.(1.2) and (1.3) is the validity of the following important identity (see [Y-O.] Eqs.(9.2) and (10.2))

\[
K(u,v)L(x,y) = \epsilon \delta L(K(u,v)x,y) - \epsilon L(K(u,v)y,x) (1.4)
\]

\[
L(u,v,L(x,y)) = L(L(u,v)x,y) - \delta L(u,v,K(x,y)v) (1.5)
\]

We note that (-1,1)FKTS is said to be a generalized Jordan triple system of second order [12] in section 2, becaue that is a generalization of concept of Jordan triple systems.

We can then construct a Lie algebra for \( \delta = +1 \) and a Lie superalgebra for \( \delta = -1 \) as follows:

Let \( W \) be a space of \( 2 \times 1 \text{matrices over } \mathbb{V} \)

\[
W \rightarrow \mathbb{V}
\]

and define a tri-linear product:

\[
W \otimes W \otimes W \rightarrow W \text{ by }
\]

\[
\begin{bmatrix}
L(x_1,y_1) - \delta L(x_2,y_1) \\
\epsilon L(y_1,x_1) - \epsilon \delta L(x_1,x_2)
\end{bmatrix}
\]

(1.6)

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Then, it defines a Lie triple system for $\delta = +1$ and an anti-Lie triple system for $\delta = -1$. We then note

$$\hat{L} = \text{span}\left\{ \frac{\partial L(x,y)}{\partial K(z,w)} \right\} \mid x,y,z,w,u,v \in V \right\} (1.7)$$

is a Lie subalgebra of $\text{Mat}(\text{End}(V))$ where $B$ for an associative algebra $B$ implies a Lie algebra with bracket; $[x,y] = xy - yx$. We note also then

$$\hat{D} = \left\{ \frac{\partial L(x,y)}{\partial K(z,w)} \right\} \in L(W, W)^{(1.8)}$$

is a derivation of the triple system. Setting

$$L_\delta = \text{span}\left\{ X = \left( \begin{array}{c} x \\ y \end{array} \right) \mid x,y \in V \right\} (1.9)$$

then $L$ defined by

$$L = L_+ \oplus L_-(1.10)$$

gives a Lie algebra for $\delta = +1$ and a Lie superalgebra for $\delta = -1$ where

$$L_\delta = \{ D \mid D \text{ is a derivation of } L \}, (1.11)$$

i.e., $D$ satisfies

$$= D[X_1, X_2, X_3]$$

and hence induces also

$$[D[X_1, X_2, X_3]], (1.12a)$$

if we define the bracket by

$$[D_1 \oplus X_1, D \oplus X_2, J] = ([D_1, D] + L(X_1, X_2)) \oplus (D_1, X_2, D X_3), (1.13)$$

where

$$[D_1, D_2] = D_1 D_2 - D_2 D_1$$

and

$$L(X_1, X_2) = [X_1, X_2] = \left[ \begin{array}{c} x_1 \\ y_1 \\ z_1 \end{array} \right] \left[ \begin{array}{c} x_2 \\ y_2 \\ z_2 \end{array} \right]$$

$$= \left( \begin{array}{c} x_1 \\ y_1 \\ z_1 \end{array} \right) \left( \begin{array}{c} x_2 \\ y_2 \\ z_2 \end{array} \right)$$

$$= \left( \begin{array}{c} x_1 + \delta L(x_1, y_1) \\ y_1 + \delta L(y_1, x_1) \\ z_1 + \delta L(z_1, x_1) \end{array} \right) (1.14)$$

Note that the endomorphism $L(X,Y)$ is then an inner derivation of the triple system.

Since $L_+ \supset L_+$, we shall mainly discuss a subsystem $\hat{L}$ of $L$, given by rather than the larger $L$. Then, $\hat{L}$ is 5-graded

$$\hat{L} = L_2 \oplus L_1 \oplus L_0 \oplus L_{-1} \oplus L_{-2}(1.16)$$

where

$$L_2 = \text{span}\left\{ \left[ \begin{array}{c} 0 \\ -\epsilon K(x,y) \end{array} \right] \mid x,y \in V \right\} (1.17a)$$

$$L_1 = \text{span}\left\{ \left[ \begin{array}{c} 0 \\ x \end{array} \right] \mid x \in V \right\} (1.17b)$$

$$L_0 = \text{span}\left\{ \left[ \begin{array}{c} L(x,y) \\ 0 \\ 0 \end{array} \right] \mid x,y \in V \right\} (1.17c)$$

$$L_{-1} = \text{span}\left\{ \left[ \begin{array}{c} x \\ 0 \\ 0 \end{array} \right] \mid x \in V \right\} (1.17d)$$

$$L_{-2} = \text{span}\left\{ \left[ \begin{array}{c} 0 \\ 0 \\ \delta K(x,y) \end{array} \right] \mid x,y \in V \right\} (1.17e)$$

Here, we utilized the following Proposition for some of its proof.

**Theorem 1:** ([K-O.], [K-M.O.]) Let $(V, (xyz))$ be a $(\epsilon, \delta)$-Freudenthal-Kantor triple system with an endomorphism $P$ such that $P^\epsilon = -\epsilon \text{Id}$ and $P(xyz) = (PxPyPz)$ Then, $(V,[xyz])$ is a Lie triple system (for $\delta=1$) and anti-Lie triple system (for $\delta=-1$) with respect to the product

$$[xyz] = (xPyz) - \delta(yPzx) + \delta(xPzy) - (yPzx).$$

In passing, we note that the standard $\hat{L} = \Sigma_{\epsilon, \delta} \hat{L}_\delta$ is a result of Theorem 1 immediately with

$$P = \left[ \begin{array}{c} 0, \delta \\ -\epsilon, 0 \end{array} \right] \text{ and } x \to X \text{ etc.}$$

Next, we introduce $\theta, \sigma(\lambda) \in \text{End}(\hat{L})$ for any $\lambda \in F (\lambda \neq 0)$, being non-zero constant by

$$\theta \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} -\epsilon x \\ \delta y \end{array} \right) (1.18a)$$

$$\sigma(\lambda) \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} \lambda x \\ \frac{1}{\lambda} y \end{array} \right) (1.18b)$$

in $W = L_2 \oplus L_{-1} \oplus L_{-2}$. We may easily verify that they are automorphism of $[W,W,W]$, i.e., we have for example

$$\theta \left( [X,Y,Z] \right) = (\theta X, \theta Y, \theta Z)$$

for $X,Y,Z \in W$. We then extend their actions to the whole of $\hat{L}$ in a natural way to show that they will define automorphism of $\hat{L}$. They moreover satisfy

(i) $\sigma$

$$\sigma(1) = \text{Id}, \theta \sigma = \text{Id}(1.19a)$$

where $\text{Id}$ is the identity mapping

(ii) $\theta^\epsilon = -\epsilon \text{Id}$ for $L_2$ but $\theta^\delta = \text{Id}$ for $L_{-1}(1.19b)$

(iii) $\sigma(\mu)\sigma(\nu) = \sigma(\mu \nu)$ for $\mu, \nu \in F, \mu \nu \neq 0$

(iv) $\sigma(\lambda) \theta \sigma(\lambda) = \theta$ for any $\lambda \in F (\lambda \neq 0). (1.19d)$

We call the group generated by $\sigma(\lambda)$ and $\theta$ satisfying these conditions simply as $D(\epsilon, \delta)$ and $P(xyz) = (PxPyPz)$. Then, $(V,[xyz])$ is a Lie triple system defined by $[xyz] = (xPyz) - \delta(yPzx) + \delta(xPzy) - (yPzx)$.

Conversely any 5-graded Lie algebra (or Lie superalgebra) with such automorphism $\theta$ and $\sigma(\lambda)$ satisfying Eqs.(1.19) lead essentially to a $(\epsilon, \delta)$ FKTS in $L$, with a triple product defined by $[x,y,z] = [\{x, \theta y, z\}]$ for $x,y,z \in F [L_2, [6]]$.

We note that the corresponding local symmetry of $D(\epsilon, \delta)$ yields a derivation of $\hat{L}$, given by

$$h = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] (1.20)$$
which satisfies
\[ h[X,Y,Z] = [hX,YZ] + [X,hYZ] + [XY,hZ] \] (1.21a)
as well as \([h,[X,Y]] = [hX,Y] + [X,hY] \) (1.21b) for \(X,Y,Z \in W\).

We can find a larger automorphism group of \(\tilde{L}_n\), if we impose some additional conditions. First suppose that \(K(x,y)\) is now expressed as
\[ K(x,y) = \epsilon_1 \delta L(y,x) - \epsilon_2 L(x,y) \] (1.22)
for any \(x,y \in V\). We call then the triple system to be a special \((\epsilon, \delta)\) FKTS [13]. Moreover for the case of special \((\epsilon, \delta)\) FKTS (i.e. \(\epsilon = \delta\)), the automorphism group of \(\tilde{L}_n\) turns out to be a larger \(SL(2,F) (= Sp(2,F))\) group which contains \(D(\epsilon,\delta)\) as its subgroup. In this case, the triple system \([W, W, W]\) becomes invariant under
\[ U(x,y) \rightarrow U(x,y)^{(1,2a)} \]
for any 2x2 \(SL(2,F)\) matrix \(U\).

Also, the associated Lie algebras or superalgebras are BC\(_\epsilon\)-graded algebra of type C\(_\delta\).

Finally, we consider a ternary system \((V, xy, xyz)\) (\(V_{xy}xyz\)) where \(xy\) and \(xyz\) are binary and ternary products, respectively, in the vector space \(V\). Suppose that they satisfy

1. The triple system \((V_{xy}xyz)\) is a \((-1,1)\) FKTS.
2. The binary algebra \((V_{xy})\) is unital and involutive \((\overline{xy} = yx)\) with the involutive map \(U_{\chi} = (\chi(x-y)) + (\chi^2(x+y)) \) (1.23)
3. The triple product \(xyz\) is expressed in terms of the bi-linear products by
\[ xyz = (\overline{xy})x - (\overline{yx})y + (xy^2)z \] (1.24)

We may call the ternary system \((V_{xy}xyz)\) to be Allison-ternary algebra or simply \(\tilde{A}\)-ternary algebra, since \(\tilde{A} = (V_{xy})\) is then the structurable algebra [14-16]. This case is of great interest, first because structurable algebras exhibit a trinity relation [16,17], and second because we can construct another type of Lie algebras independently of the standard construction of \((-1,1)\) FKTS, which is \(S_\epsilon\)-invariant and of BC\(_\delta\) graded Lie algebra of type \(B_\delta\). The relationship between the Lie algebra constructed in the new way and that given as in Eq.(1.17) is by no means transparent. Note that the group \(D(-1,1)\) contains \(S_\epsilon\) but not \(S_\delta\) symmetry. We may show that if the field \(F\) contains the square root \(\sqrt{-1}\), then Eq.(1.17) can be prolonged to yield the Lie algebra for the structurable algebra.

The Lie algebras constructed as in Eqs.(1.13) and (1.14) is also a \(BC_\epsilon\)-graded Lie algebra of type \(B_\delta\) without assuming \(\sqrt{-1} \in F\). (6). Also, if \(F\) is an algebraically closed field of characteristic zero, then any simple Lie algebra is known to be \(S_\delta\)-invariant and can be constructed by some structurable algebra, so that any such Lie algebra is also a \(BC_\epsilon\)-graded Lie algebra of type \(B_\delta\) (as well as of type \(C_\delta\)). Of course, the underlying \(sl(2)\) symmetry is different for both \(B_\delta\) and \(C_\delta\) cases. Roughly speaking, it seems that these concept are a version of Lie algebra theory corresponding with a Galois algebra of Lie algebras numbers theory.

The contents of this section is a cowork with Prof. Okubo, and the details will be discussed in other papers.

Hermitian triple systems

For a geometrical object based on triple systems in this section, first we note that the symmetric bounded domains are a one to one, correspondence to hermite Jordan triple systems, such that a certain trace form is positive definite hermite. Hence as a generalization of these triple systems, we are interesting to investigate for structure theory of hermitian generalized Jordan triple systems, in particular, the case of hermitian generalized Jordan triple systems of second order (i.e. hermitian \((-1,1)\) -Freudenthal-Kantor triple system ).

We shall concern with algebras and triple systems which are infinite over a complex number field, unless otherwise specified in this section.

**Definition:** A triple system \(V\) is said to be a \(*\)-generalized Jordan triple system of second order if relations (0)—(iv) satisfy;

0) \(V\) is a Banach space,

i) \([L(a,b),L(c,d)] = L(abc,d) - L(c,bad),\)

ii) \(K(<abc>,d) + K(c, <abd>) + K(a,K(c,d)b) = 0,\)

where \(L(a,b,c) = <abc>\) and \(K(a,b,c) = a<cb> - b<ac>\),

iii) \(<xyz>\) is C-linear operator on \(x, y\) and C-anti-linear operator on \(z\),

iv) \(<abc>\) continuous with respect to a norm \(\|x\|\) that is, there exists \(K > 0\) such that
\[ \|<xxx>\| \leq K \|x\|^3 \text{for all} x \in V.\]

Furthermore a \(*\)-generalized Jordan triple system of second order is said to be hermitian if it satisfies the following condition, v) all operator \(L(x,y)\) is a positive hermitian operator with a hermitian matrix
\[(x,y) = \text{trace} L(x,y), (2.1)\]
that is, \((L(xy)z,w) = (\overline{z},L^*(xy)w)\), and \((x,y) = \overline{(y,x)}\). In particular, if a triple system \(V\) satisfies the condition (i), (ii), (iii), (iv) and (v), then it is said to be a hermitian \(*\)-generalized Jordan triple system.

Furthermore, as a generalization of the generalized Jordan triple system of second order, it is said to be a \(\epsilon,\delta\)-Freudenthal-Kantor triple system if the following relations satisfy:

i) \([L([a,b],[c,d]),d] = L(abc,d) - L(c,bad),\)

ii) \(K(<abc>,d) + K(c,<abd>) + \delta K(a,K(c,d)b) = 0,\)

where \(K(a,b,c) = <abc>\) and \(K(a,b,c) = a<cb> - b<ac>\),

\[ K(x,y,z) = <xyz> - y<xyz> = 0 \]

(identically zero).

**Example 1:** Let \(V\) be a \(*\)-triple (for the definition, to Loos [3]). Then \(V\) is a hermitian \(*\)-generalized Jordan triple system of second order, because they satisfy the condition (i) and a special case of (ii), that is,

\[ K(xy,z) = <xyz> - y<xyz> = 0 \]

(Identically zero).

**Example 2:** Let \(T^*_{\epsilon,\delta}\), be the space of diagonal matrix of \(n \times n\) with element of the complex number. Then \(T^*_{\epsilon,\delta}\) is a \(*\)-generalized Jordan triple system of second order, with respect to the product and the norm
\[ <xyz> = x\overline{T}^2z + \overline{y}T^2x - \overline{T}^2z \]
for all \( \lambda \in \mathbb{R} \) and \( e_i \) are matrix unit elements of \( T_{n \times n} \) and \( x^T \) is the transpose of \( x \).

Next let \( V \) be a *-generalized Jordan triple system. Then we may define the notion of a tripotent and a bitripotent as follows.

**Definition:** It is said to be a tripotent of \( V \) if
\[
\langle \langle c_1, c_2 \rangle \rangle = -1/2c_1 < c_1, c_2 \rangle < = -1/2c_1,
\]
and other products are zero.

**Definition:** It is said to be a bitripotent of \( V \) if a pair \( (c_1, c_2) \) of the tripotents satisfy the relations
\[
\langle \langle c_1, c_2 \rangle \rangle = \alpha c_1, \langle c_1, c_2 + c_2, c_1 \rangle = \gamma c_1,
\]
and other products are zero.

Then we may get
\[
\|x\| = \max \{\lambda_j, \mu_j \} \quad \text{for } x = \Sigma \lambda_j E_{ij},
\]
where the notations are denoted by \( \lambda_i, \mu_i \) is 1 and other element is zero. \( E_i \) and \( \sqrt{E_i} \) are positive definite. It is enough to show the condition (iv). By means of result of the property of matrix, we can write
\[
x = \Sigma \lambda_i e_i, \quad \|x\| = \max \{\mu_j \},
\]
e_1 are tripotents or bitripotents if \( e_i \) is the unit element of matrix.

Moreprecisely speaking, we have
\[
\|x\| = \max \{\lambda_j, \mu_j \} \quad \text{for } x = \Sigma \lambda_j E_{ij} + \mu_i \sqrt{E_i},
\]
where the notations are denoted by \( \lambda_i, \mu_i \) is 1 and other element is zero. \( E_i \) and \( \sqrt{E_i} \) are bitripotents.

Furthermore, we note \( (E_i, \sqrt{E_i}) \) are bitripotents.

Then we have
\[
\|x + y\| \leq \|x\| + \|y\|.
\]

These show that \( D_{nk}^* \) is a hermitian *-generalized Jordan triple system of second order, that is, a hermitian (1,11)-FKTS.

**Examples 4:** Let \( S_{n,k}^* \) be the set of all \( 2n \times k \) matrices with operation
\[
< XYZ := X(\overline{Y}) + Z(\overline{Y}) > X \in S_{n,k}^* \quad (2.4)
\]
then \( S_{n,k}^* \) is a hermitian *-generalized Jordan triple system of second order.

In fact, the elements
\[
\Sigma (\lambda_i^{(1)})^2 + \mu_i^{(1)} \sqrt{E_i^{(1)}} + \Sigma (\lambda_i^{(2)})^2 + \mu_i^{(2)} \sqrt{E_i^{(2)}}
\]
and the norm is defined by
\[
\|X\| = \max \{\lambda_j^{(1)} |, |, \lambda_j^{(2)} |, |, \mu_j^{(1)} |, |, \mu_j^{(2)} \}
\]
where \( E^{(1)} \) is the matrix unit of \( S_{n,k}^* \) and \( E^{(2)} \) is the matrix unit of \( S_{n,k}^* \).

By straightforward calculations as well as Example 2.3, we can show that \( S_{n,k}^* \) is a hermitian *-generalized Jordan triple system of second order.

**Example 4:** Let \( A_{\lambda}^{(1)} \oplus A_{\mu}^{(1)} A_{\lambda}^{(2)} \) be the set fo all pairs \( X = \langle x_i \rangle \), where \( x_i \) is a \( n \times k \) matrix and \( X \) is a \( n \times l \) matrix with operation given by formula
\[
< XYZ := X(\overline{Y}) + Z(\overline{Y}) > X \in S_{n,k}^* \quad (2.4)
\]
then \( A_{\lambda}^{(1)} \oplus A_{\mu}^{(1)} A_{\lambda}^{(2)} \) is a hermitian *-generalized Jordan triple system of second order.

In fact, for \( X = \langle x_i \rangle \), we have
\[
< X, Y > = \text{trace}(X, Y) - \text{trace}(x_i \overline{y}_i) + \text{trace}(x_i \overline{y}_i) \quad \text{hence it follows that the trace form } (, ) \text{ is positive definite.}
\]

For any element of \( A_{\lambda}^{(1)} \oplus A_{\mu}^{(1)} \), we may represent as follows;
\[
X = \Sigma (\lambda_i^{(1)})^2 + \mu_i^{(1)} \sqrt{E_i^{(1)}} + \Sigma (\lambda_i^{(2)})^2 + \mu_i^{(2)} \sqrt{E_i^{(2)}}
\]
and $(E^{(1)}_{\mu}, E^{(2)}_{\lambda}) \in A^{(1)}_\mu \oplus A^{(2)}_\lambda$ is a tripotent.

Furthermore, the norm is defined by
\[ \|X\| := \sqrt{\max \{\|X^{(1)}_{\mu}\|^2, \|X^{(2)}_{\mu}\|^2, \|\mu^{(1)}_{\mu}\|^2, \|\mu^{(2)}_{\mu}\|^2\}}. \]

**Example 6:** Let $C^{+}_m \oplus A^{+}_n$, be the set of all pairs $X = \left(\begin{array}{c} x \\ y \end{array}\right)$, where $x$ and $y$ are tripotents and also $\delta$-generalized Jordan triple system of second order.

In fact, we put $c_{ij} := \delta$ -element is $(j, i)$ -element is $1$ and zero is $1$s $(1 \leq i \leq n)$, further more $e_{kj} := (0, - , , 1, \ldots, -0)T(k \text{ the element is }1)$. Then $C^{+}_m \oplus A^{+}_n$ are tripotents and also $C^{+}_m \oplus A^{+}_n$ are tripotents but these are not bitripotents. For any element of $C^{+}_m \oplus A^{+}_n$, we may represent as follow:

\[ X = \sum \left(\begin{array}{c} \lambda^{(1)}_{\mu} \\ \mu^{(1)}_{\mu} \end{array}\right) \delta^{1}_{\mu} \sqrt{\|c_{\mu}\|} + \sum \left(\begin{array}{c} \lambda^{(2)}_{\mu} \\ \mu^{(2)}_{\mu} \end{array}\right) \delta^{2}_{\mu} \sqrt{\|c_{\mu}\|} \]

and the norm is defined by
\[ \|X\| := \sqrt{\max \{\|X^{(1)}_{\mu}\|^2, \|X^{(2)}_{\mu}\|^2, \|\mu^{(1)}_{\mu}\|^2, \|\mu^{(2)}_{\mu}\|^2\}}. \]

In the end of this section, we note a Peirce decomposition as follows ([12] for the case of $\delta = 1$).

**Theorem 1:** For $\delta = \pm 1$ and hermitian $(-1, \delta)$ Freudenthal-Kantor triple system $V$ with a tripotent $C$ s.t. $\langle c, c \rangle = c$ we have

\[ V = V_{1} \oplus V_{2} \oplus V_{3} \]

where denoted by $V_{1} = \{x | N_{x} = 0, T_{x} = 0\}$, $V_{2} = \{x | N_{x} \neq 0, T_{x} = 0 \}$, and denoted by $R(x, y)z = <xyz> + <zyx>$, $N_{x} = L(c, c) - \delta R(c, c)$ and $T_{x} = (1 + \delta)L(c, c) - R(c, c) + Id.$

This section is a cowork with Dr M.Sato with an application to physics and the details will be considered in future paper [18].

**Concluding Remark**

In this section, we shall briefly describe a correspondence with the triple systems and the Lie algebras or superalgebras of simple classical type associated with their triple systems [1,2,7,10,11].

Let $V$ be the matrix set of $\mathsf{Mat}(m, n; F)$ and the triple product is defined by

\[ <xyz> = x_{1}y_{1}z_{1} + \delta (z_{1}x_{1}y_{1} - x_{1}y_{1}z_{1}) \quad x, y, z \in V. \]

Then there exists 4 cases of $(-1, \delta)$-Freudenthal-Kantor triple systems with $\mu = 0$ or 1 and $\delta = \pm 1$.

The standard embedding Lie algebras or superalgebras $L = L(\mathbb{W}, W) \oplus W$, where $W = V \oplus V$ (cf. section 1) associated with the triple product $\langle \cdot, \cdot, \cdot \rangle$ are appeared by 4 types as follows;

(i) 5 graded Lie algebra $(\delta = 1, \mu = 1)$

\[ B_{m,n} = so(2m+2l+1) (n = 2l+1), D_{m,n} = so(2m+2l) (n = 2l). \]

(ii) 3 graded Lie algebra $(\delta = 1, \mu = 0)$

\[ A_{m,n} = sl(n+m). \]

(iii) 5 graded Lie superalgebra $(\delta = -1, \mu = 1)$

\[ . . . B(l,m) = osp(2l+1|2m) (n = 2l+1), D(l,m) = osp(2l|2m) (n = 2l). \]

In particular, if $m = 1$, then the triple product

\[ <xyz> = <x,y,z> - <z,y,x> + <z,x,y> \]

is a $(-1, -1)$-Freudenthal-Kantor triple system satisfying $K(x,z) = <xyz> + <zyx> = 2 <x,z,y>$, and so dim(\{x,z\} span) = 1 (called a balanced triple system), where $<x,y> = x_{1}y_{1} + \ldots + x_{n}y_{n}$.

(iv) 3 graded Lie superalgebra $(\delta = -1, \mu = 0)$

\[ A(m,n-1) = sl(m|n) (m \neq n), A(m-1|m-1) = psd (m|m) (m = n). \]

**Reference**