Keywords: Analytic functions; Struve functions; Bessel functions; Integral operators

Introduction and Preliminaries

Let consider $U$ the unit disc. Let $H(U)$ be the set of holomorphic functions in the unit disc $U$ [1-9].

Consider $A=\{f \in H(U): f(z)=z+a_2z^2+a_3z^3+... \}$ be the class of analytic functions in $U$ and $S=\{f \in A: f \text{ is univalent in } U \}$.

Theorem 1.1
If the function $f$ is regular in unit disc $U$, $f(z)=z+a_2z^2+...$ and
\[
\left(1-|z|^2\right)\frac{f''(z)}{f'(z)} < 1
\]
for all $z \in U$, then the function is univalent in $U$ [1].

Theorem 1.2
If the function $g$ is regular in $U$ and $|g(z)|<1$ in $U$, then for all $\xi \in U$ and $z \in U$ the following inequalities hold [4]
\[
\frac{g(z)-g(\xi)}{1-g(z)g(\xi)} > \frac{z-\xi}{1-|z|^2}
\]
and
\[
g'(z) \leq \frac{1-g'(z)\xi^2}{1-|z|^2}
\]
the equalities hold in case $g(z)=e^{z+u}\frac{1}{1+re^z}$ where $|e|=1$ and $|u|<1$.

Remark 1.1
For $z=0$ from inequality (2) we obtain for every $\xi \in U$ [2]
\[
\frac{g(z)-g(0)}{1-g(0)g(z)} \leq |\xi|
\]
and hence
\[
|g'(\xi)| \leq \frac{|\xi|+|g(0)|}{1+|g(0)||\xi|}
\]
Considering $g(0)=a$ and $\xi=z$, then
\[
|g(z)| \leq \frac{|z|+|a|}{1+|a||z|}
\]
for all $z \in U$.

Let us consider the second-order inhomogeneous differential equation (((10)), p.341)
\[
z^2w''(z) + zw'(z) + \left(z^2 - v^2\right)w(z) = \frac{4\left(z^2\right)^{v+1}}{\sqrt{4\left(z^2\right) + 1}}
\]
whose homogeneous part is Bessel’s equation, where $v$ is an unrestricted real (or complex) number. The function $H_v$, which is called the Struve function of order $v$, is defined as a particular solution of (7). This function has the form
\[
H_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(v+n+\frac{3}{2}\right)} \left(z^2\right)^n
\]
for all $z \in \mathbb{C}$ (8)

We consider the transformation
\[
g_v(z) = 2\sqrt{\pi} \Gamma\left(v+\frac{3}{2}\right) z^{\frac{v+1}{2}} H_v\left(z^{\frac{1}{2}}\right)
\]
(9)

After some calculus we obtain
\[
g_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(v+n+\frac{3}{2}\right)} z^n
\]
(10)

Using Theorem 2.1 ([5]) for our case with $b=c=1$, $k=v+\frac{3}{2}$ we obtain that:

Theorem 1.3
[5, [3], if $v > \sqrt{3}\frac{7}{8}$ then the function $g_v$ is univalent in $U$.

The Bessel function of the first kind is defined by
\[
J_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma\left(n+v+1\right)} \left(z^2\right)^n
\]
(11)

We consider the transformation

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Received May 24, 2017; Accepted August 02, 2017; Published August 13, 2017


doi: 10.4172/2168-9679.1000366

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J Appl Comput Math, an open access journal
ISSN: 2168-9679

Volume 6 • Issue 3 • 1000366
\[ f_i(z) = 2\Gamma(1+v) z^{-2} f_i(\sqrt{z}) \]

After some calculus we obtain
\[ f_i(z) = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n \Gamma(1+v) \left(n + v + 1\right)^4 z^n \]

**Theorem 1.4**

If \( v > -2 \) then \( f_i(z) \in U(0, (v+2)) \) and \( f_i \) is univalent in \( U(0, (v+2)) \) [7,9].

**Main Results**

**Theorem**

Let \( f_i \) be Bessel functions, \( z \in U, v \in (-2, 1) \), \( \alpha_i \in \mathbb{C} \) where

\[ f_i(z) = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n \Gamma(1+v) \left(n + v + 1\right)^4 z^n \]

If

\[ \left| \frac{\partial f_i}{\partial z}(z) \right| < 1, \text{ for all } i \in [1, 2, ..., n], \ (\forall) z \in U \]

\[ \left| \alpha_i + \alpha_i^2 + ... + \alpha_i^n \right| < 1 \]

\[ \left| \alpha_i, \alpha_i^2, ..., \alpha_i^n \right| < 1 \]

By using the relations (14) and (15) we obtain \(|h(z)| < 1\) and

\[ h(0) = \left| \frac{1}{\alpha_i \alpha_i^2 + ... + \alpha_i^n} \right| \]

\[ \left| \alpha_i \alpha_i^2 + ... + \alpha_i^n \right| < 1 \]

Applying Remark 1.1 for the function \( h \) we obtain

\[ \left| 1 - |z|^2 \right| F''(z) F'(z) < \left| \alpha_i, \alpha_i^2, ..., \alpha_i^n \right| \left| 1 - |z|^2 \right| \left| \frac{1}{1 + |z|^2} \right| \]

for all \( z \in U \).

Let's consider the function \( H(x) = (1-x^2)^{1/2} \) for all \( x \), \( x \in [0, 1] \)

\[ H(x) = (1-x^2)^{1/2} \frac{1}{1 + \left| x \right|^2} \]

\[ H\left( \frac{1}{2} \right) = 3 \left( \frac{1}{2} \right)^{1/2} \]

We obtain

\[ \left(1 - |z|^2\right)^2 F''(z) F'(z) < \left| \alpha_i, \alpha_i^2, ..., \alpha_i^n \right| \left| 1 - |z|^2 \right| \left| \frac{1}{1 + |z|^2} \right| \]

Applying the condition (16) we obtain:

\[ \left(1 - |z|^2\right)^2 F''(z) F'(z) < 1 \]

and from Theorem 1.1 then \( F \in S \).

For \( \alpha_i, \alpha_i^2, ..., \alpha_i^n \) in Theorem 2.1 we obtain the next corollary:

**Corollary 2.1**

Let \( f_i \) be Bessel functions, \( z \in U, v \in (-2, 1) \), \( \alpha_i \in \mathbb{C} \) where

\[ f_i(z) = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n \Gamma(1+v) \left(n + v + 1\right)^4 z^n \]

If

\[ \left| \frac{\partial f_i}{\partial z}(z) \right| < 1, \text{ for all } i \in [1, 2, ..., n], \ (\forall) z \in U \]

\[ \left| \alpha_i + \alpha_i^2 + ... + \alpha_i^n \right| < 1 \]

\[ \left| \alpha_i, \alpha_i^2, ..., \alpha_i^n \right| < 1 \]

\[ \left| \alpha_i \alpha_i^2 + ... + \alpha_i^n \right| < 1 \]

Then

\[ F(z) = \int_a^b \left( \frac{f_i(t)}{t} \right)^{a_i} \left( \frac{f_i(t)}{t} \right)^{a_i^2} \left( \frac{f_i(t)}{t} \right)^{a_i^n} dt < 1 \]

\[ \left| 1 - |z|^2 \right| F''(z) F'(z) < 1 \]

\[ \left| \alpha_i, \alpha_i^2, ..., \alpha_i^n \right| \left| 1 - |z|^2 \right| \left| \frac{1}{1 + |z|^2} \right| \]

**Theorem 2.2**

Let \( g_i \) be Struve functions, \( z \in U, v \in (-2, 1) \), \( \alpha_i \in \mathbb{C} \) where
For $z=0$ we have
\[
g(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \frac{3}{2} + \frac{1}{n + 3/2} \right) z^n.
\]

Consider the function
\[
h(z) = \frac{g(z)}{z^{\alpha}} G(z).
\]

The function $h$ has the form:
\[
h(z) = \frac{1}{\alpha_1, \alpha_2, \ldots, \alpha_n} \frac{g(z)}{z^{\alpha}} G(z).
\]

We have:
\[
h(0) = \frac{1}{\alpha_1, \alpha_2, \ldots, \alpha_n} b^2 + \ldots + \frac{1}{\alpha_1, \alpha_2, \ldots, \alpha_n} b^2
\]

Where
\[
b_i^2 = \frac{1}{15(2v_i + 3)(2v_i + 5)}
\]

Then, applying Remark 1.1 for the function $h$ we obtain
\[
1 + \left| h(z) \right| < \frac{H(z)}{G(z)}.
\]

Let's consider the function $H : [0,1] \to \mathbb{R}$
\[
H(x) = (1-x^2)x^{\alpha_1} + x^{\alpha_2} > 0
\]

Then we obtain
\[
1 - \left| h(z) \right| < \frac{H(z)}{G(z)}.
\]

And from Theorem 1.1 then $G \in S$.

In Theorem 2.2 we consider $\alpha_1, \alpha_2, \ldots = \alpha_n$, and obtain the next corollary:

Corollary 2.2 Let $g_n$, Bessel functions, $z \in U, \nu \in (2,1), \alpha \in C$ where
\[
g_n(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \frac{3}{2} + \frac{1}{n + 3/2} \right) z^n.
\]

If
\[
\left| \frac{g_n'(z) - g_n(z)}{g_n(z)} \right| \leq 1 , \text{ for all } i \in [1, \ldots, n], \forall z \in U
\]

Then
\[
\max \left| \left( 1 - \left| h \right| \right) \frac{H(z)}{G(z)} \right| \leq 1.
\]

References