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Characterizations of Certain Doubly Truncated Distribution Based on Order Statistics

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Abstract

In this paper, we characterize doubly truncated classes of absolutely continuous distributions by considering the conditional expectation of functions of order statistics. Specific distributions considered as a particular case of the general class of distributions are Weibull, Pareto, Power function, Rayleigh and Inverse Weibull.

Keywords: Double truncated; Order statistics; Conditional expectation; Weibull, Pareto; Rayleigh; Inverse Weibull distributions

AMS 2000 Subject Classification: 62E10; 62G30

Introduction

The order statistics arise naturally in many real life applications and it is considered as an increasingly important subject. Articles relating to this area have appeared in numerous different publications. Many authors have studied order statistics; for example, David [1], Balakrishnan and Cohen [2], Arnold et al. [3], David [4], David and Nagaraja [5] and Mahmoud et al. [6,7]. Several authors discussed conditional expectations, for example, Balakrishnan and Sultan [8], Mohie El-Din et al. [9], Abu-Youssef [10], Abd- El-Mougod [11], Shawky and Abu-Zinadah [12], Shawky and Bakoban [13] and Pushkarna et al. [14].

Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be the first n order statistics based on distribution with probability density function (pdf) f(x) and cumulative distribution function (cdf) F(x). Then the pdf of the r[th] order statistics, $X_{r:n}$, $1 \le r \le n$, is given by (see David (1981))

$$f_r(x) = C_r\left[F(x)\right]^{r-1}\left[1-F(x)\right]^{n-r} f(x), \tag{1.1}$$

where $C_r = \dfrac{n!}{(r-1)!(n-r)!}, -\infty < x < \infty,$

and the joint pdf of two order statistics $X_{r:n}$ and $X_{s:n}$, $1 \le r \le s \le n$ is given by

$$f_{r,s}(x,y) = C_{r,s}\left[F(x)\right]^{r-1}\left[F(y)-F(x)\right]^{s-r-1}\left[1-F(y)\right]^{n-s} f(x) f(y), -\infty < x < y < \infty, \tag{1.2}$$

where $C_{r,s} = \dfrac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

The doubly truncated case of a distribution is the most general case since it includes the right truncated, left truncated and non-truncated distributions as special cases, Joshi [15], Balakrishnan and Joshi [16], Khan and Ali [17] and Ahmad [18], among others, investigated doubly truncated distributions.

Suppose that the random variable X has a cdf F(x) and pdf f(x), where $\alpha \le x$ β. Let, for given ε and

$$\int_\alpha^\varepsilon f(x)dx = F(\varepsilon) = P \text{ and } \int_\alpha^\gamma f(x)dx = F(\gamma) = Q.$$

Then the doubly truncated pdf of X, say g(x), and cdf, say G(x), are given respectively by

$$g(x) = \frac{f(x)}{l_1}, \alpha \le \varepsilon < x < \gamma \le \beta, \tag{1.3}$$

$$G(x) = \frac{F(x)-P}{l_1}, \varepsilon < x < \gamma, \tag{1.4}$$

where

$l_1 = Q - P$, $G(\varepsilon)=0$ and $G(\gamma)=1$.

The conditional density function of $X_{s:n}=y$, given that $X_{r:n}=x$ is given [3] by

$$f_{s|r}(y|X_{r:n}=x) = \frac{(n-r)!}{(s-r-1)!(n-s)![1-G(x)]^{n-r}}.$$
$$[G(y)-G(x)]^{s-r-1}[1-G(y)]^{n-s} g(y),$$
$$\varepsilon < x < y < \gamma. \tag{1.5}$$

Also, the conditional density function of $X_{r:n}=x$, given that $X_{s:n}=y$ is *given by*

$$f_{r|s}(x|X_{s:n}=y) = \frac{(s-1)!}{(r-1)!(s-r-1)![G(y)]^{s-1}}. \tag{1.6}$$
$$[G(x)]^{r-1}[G(y)-G(x)]^{s-r-1} g(x)$$

Let

$$\mu_{s|r} = E[\varphi(X_{s:n})|X_{r:n}=x] \text{ and } \mu_{r|s} = E[\varphi(X_{r:n})|X_{s:n}=y],$$

where $\varphi(.)$ is a monotonic, continuous and differentiable function on the interval (α, β). For abbreviation, we will denote

$$\mu_{s|r} = E_{s|r}[\varphi(Y)|X_{r\,n}=x] \text{ and } \mu_{r|s} = E_{r|s}[\varphi(X)|X_{s:n}=y]. \tag{1.7}$$

Main Results

In this section, we characterize three general classes of distributions,

$$F(x) = 1-[b-a e^{-\varnothing(x)}]^c, \alpha < x < \beta, \text{ i.e.,}$$

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$$G(x)=\frac{1}{l}\{\left[b-a\,e^{-\varnothing(x)}\right]^c-v\},\varepsilon<x<\gamma, \quad (2.1)$$

where

v=1−P, l=P − Q, P=$F(\varepsilon)$, Q=$F(\gamma)$, $G(\varepsilon)$=0, $G(\gamma)$=1,

$F(\alpha)$=0, $F(\beta)$=1, and $a \neq 0$, c 0, b are finite constants.

$F(x)=1-[a-b\,\varnothing(x)]^c$, $\alpha<x<\beta$, i.e.,

$$G(x)=\frac{1}{l}\{\left[a-b\varnothing(x)\right]^c-v\},\varepsilon<x<\gamma, \quad (2.2)$$

where

$v=1-P, l=P-Q, P=F(\varepsilon), Q=F(\gamma), G(\varepsilon)=0, G(\gamma)=1,$

$F(\alpha)$=0, $F(\beta)$=1, and $b \neq 0$, $c \neq 0$, a are finite constants.

$F(x)=1-[b-a\,e^{-C\varnothing(x)}],\alpha<x<\beta$, i.e.,

$$G(x)=\frac{1}{l}\{[b-a\,e^{-C\varnothing(x)}]-v\},\varepsilon<x<\gamma, \quad (2.3)$$

where

v=1 − P, l=P − Q, P=$F(\varepsilon)$, Q=$F(\gamma)$, $G(\varepsilon)$=0, $G(\gamma)$=1,

$F(\alpha)$=0, $F(\beta)$=1, and $C \neq 0$, $a > 0$, $b > 0$ are finite constants.

Note: If we put l=−1, v=1, thus G(x) reduces to complete cdf of x, i.e. F(x), $\alpha < x < \beta$.

Let X be an absolutely continuous random variables with pdf g(x), cdf G(x) and $\varnothing(x)$ is a monotonic, continuous and differentiable function on (ε,γ).

Theorems 1-4 given below characterize the general class given by (2.1), Theorems 5-8 characterize the general class given by (2.2), while Theorems 9-12 characterize the general class given by (2.3).

Theorem 1

Referring to (1.6), (1.7) and (2.1), then

$$\mu_{r|s}=\mu_{r+1|s}+\frac{1}{acr}\left\{aE_{r|s}\left[V(X)\middle|X_{s:n}=y\right]-bE_{r|s}\left[V(X)e^{\varphi(X)}\middle|X_{r:s}=y\right]\right\}, \quad (2.4)$$

where

$$V(x)=\frac{lG(x)}{lG(x)+v}. \quad (2.5)$$

Proof

It is clear from (1.6) and (1.7) that

$$\mu_{r|s}=\frac{(s-1)!}{(r-1)!(s-r-1)![G(y)]^{s-1}}\int_{\varepsilon}^{y}\varphi(x)\left[G(x)\right]^{r-1}\left[G(y)-G(x)\right]^{s-r-1}g(x)dx. \quad (2.6)$$

Integrating (2.6) by parts, we get

$$\mu_{r|s}=\mu_{r+1|s}-\frac{(s-1)!}{r!(s-r-1)![G(y)]^{s-1}}\int_{\varepsilon}^{y}\dot{\varphi}(x)\left[G(x)\right]^{r}\left[G(y)-G(x)\right]^{s-r-1}dx. \quad (2.7)$$

Differentiating (2.1) with respect to x, we have

$$\dot{\varphi}(x)=\frac{be^{\varphi(x)}-a}{ac}\frac{lg(x)}{lG(x)+v}. \quad (2.8)$$

From (2.7) and (2.8), we obtain

$$\begin{aligned}\mu_{r|s}&=\mu_{r+1|s}-\frac{(s-1)!}{(ac)r!(s-r-1)!\left[G(y)\right]^{s-1}}\int_{\varepsilon}^{y}(be^{\varphi(x)}-a)V(x)\left[G(x)\right]^{r-1}\left[G(y)-G(x)\right]^{s-r-1}g(x)dx\\&=\mu_{r+1|s}-\frac{1}{acr}\int_{\varepsilon}^{y}\left[be^{\varphi(x)}-a\right]V(x)g_{r|s}(x|y)dx.\end{aligned} \quad (2.9)$$

Simplifying (2.9), we get (2.4). Thus, the theorem is proved.

Theorem 2

Referring to (1.6), (1.7), then (2.1) if and only if

$$\mu_{r|r+1}=\varphi(y)+\frac{1}{ac}\sum_{i=0}^{\infty}(-1)^{i+1}\left(\frac{l}{v}\right)^{i+1}\frac{[G(y)]^{i+1}}{r+i+1}\{bE_{r+i+1|r+i+2}\left[e^{\varphi(X_{r+i+1:n})}\middle|X_{r+i+2:n}=y\right]-a\}. \quad (2.10)$$

Proof

It is clear that

$$\mu_{r|r+1}=\frac{r}{[G(y)]^r}\int_{\varepsilon}^{y}\varphi(x)\left[G(x)\right]^{r-1}g(x)dx.$$

Integrating by parts, we get

$$\mu_{r|r+1}=\varphi(y)-\frac{1}{[G(y)]^r}\int_{\varepsilon}^{y}\dot{\varphi}(x)[G(x)]^r dx \quad (2.11)$$

Compensation for (2.8) in (2.11), we have

$$\mu_{r|r+1}=\varphi(y)-\frac{l}{ac[G(y)]^r}\{\int_{\varepsilon}^{y}\left(be^{\varphi(x)}-a\right)\frac{g(x)}{lG(x)+v}\left[G(x)\right]^r dx. \quad (2.12)$$

Expand $\frac{1}{lG(x)+v}$ and compensation for (2.12), after some simplification, we get (2.10). Thus (2.1) implies (2.10). Now from (1.6) and (2.10), we obtain

$$\frac{r}{[G(y)]^r}\int_{\varepsilon}^{y}\varphi(x)\left[G(x)\right]^{r-1}g(x)dx=\varphi(y)+\frac{1}{ac}\sum_{i=0}^{\infty}(-1)^{i+1}\left(\frac{l}{v}\right)^{i+1}\frac{[G(y)]^{i+1}}{r+i+1}\{b\int_{\varepsilon}^{y}e^{\varphi(x)}\left[G(x)\right]^{r+i}g(x)dx-a\}. \quad (2.13)$$

Taking the derivative, we get

$$\dot{\varphi}(y)=\frac{(\frac{l}{v})g(y)}{1+\frac{l}{v}G(y)}\left[\frac{b}{ac}e^{\varphi(y)}-\frac{1}{c}\right],$$

which gives

$$\frac{ac\dot{\varphi}(y)e^{-\varphi(y)}}{b-ae^{-\varphi(y)}}=\frac{lg(y)}{v+lG(y)}. \quad (2.14)$$

Integrate (2.14), hence G(y) has the form (2.1), and so (2.10) implies (2.1).

Special case:

Return to the (2.10), if we put l=−1, v=1we get

$$\mu_{r|r+1}=\varphi(y)+\frac{1}{ac}\sum_{i=0}^{\infty}\frac{[F(y)]^{i+1}}{i+r+1}\left\{bE_{r+i+1|r+i+2}\left[e^{\varphi(X_{r+i+1:n})}\middle|X_{r+i+2:n}=y\right]-a\right\},$$

$$\alpha<x<y<\beta \quad (2.15)$$

the relation (2.15) is before doubly truncated case.

Theorem 3

Referring to (1.5), (1.7) and (2.1), then

$$\mu_{s|r}=\mu_{s-1|r}+\frac{l}{ac(n-s+1)}\left\{aE_{s|r}\left[N(X_{s:n})\middle|X_{r:n}=x\right]-bE_{s|r}\left[e^{\varphi(X_{s:n})}N(X_{s:n})\middle|X_{r:n}=x\right]\right\}, \quad (2.16)$$

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where

$$N(y)=\frac{[1-G(y)]}{[lG(y)+v]}. \tag{2.17}$$

It is clear from (1.5) and (1.7) that

$$\mu_{s|r}=\frac{(n-r)!}{(s-r-1)!(n-s)![1-G(x)]^{n-r}}\int_x^{\gamma}\varphi(y)\left[G(y)-G(x)\right]^{s-r-1}\left[1-G(y)\right]^{n-s}g(y)dy.$$

Integrating by parts, we get

$$\mu_{s|r}=\mu_{s-1|r}-\frac{(n-r)!}{(s-r-1)!(n-s+1)![1-G(x)]^{n-r}}\int_x^{\gamma}. \tag{2.18}$$
$$\varphi'(y)[G(y)-G(x)]^{s-r-1}[1-G(y)]^{n-s+1}dy$$

Substituting (2.7) in (2.18), we get

$$\mu_{s|r}=\mu_{s-1|r}-\frac{(n-r)!}{(s-r-1)!(n-s+1)![1-G(x)]^{n-r}}\int_x^{\gamma}\frac{e^{\varphi(y)}}{ac}\left[b-ae^{-\varphi(y)}\right]N(y)[G(y)-G(x)]^{s-r-1}$$

$$\mu_{s|r}=\mu_{s-1|r}-\frac{(n-r)!}{(s-r-1)!(n-s+1)![1-G(x)]^{n-r}}\int_x^{\gamma}\frac{e^{\varphi(y)}}{ac}\left[b-ae^{-\varphi(y)}\right]N(y)[G(y)$$
$$-G(x)]^{s-r-1}\times[1-G(y)]^{n-s}dy \tag{2.19}$$
$$=\mu_{s-1|r}+\frac{l}{ac(n-s+1)}\{\int_x^{\gamma}\left[a-be^{\varphi(y)}\right]N(y)g_{s|r}(y|x)dy$$

After some simplification, we get (2.16).

Theorem 4

Referring to (1.5), (1.7), then (2.1) if and only if

$$\mu_{r+1|r}=\varphi(x)+\frac{lb}{ac(n-r)}E_{r+1|r}\left[N(y)e^{\varphi(y)}\middle|X_r=x\right]-\frac{l}{c(n-r)}E_{r+1|r}\left[N(y)\middle|X_{r:n}=x\right], \tag{2.20}$$

where $N(y)$ is defined in (2.17).

Proof

It is clear that

$$\mu_{r+1|r}=\frac{(n-r)}{[1-G(x)]^{n-r}}\int_x^{\gamma}\varphi(y)\left[1-G(y)\right]^{n-r-1}g(y)dy. \tag{2.21}$$

Integrating by parts, we get

$$\mu_{r+1|r}=\varphi(x)+\frac{1}{[1-G(x)]^{n-r}}\int_x^{\gamma}\varphi'(y)[1-G(y)]^{n-r}dy. \tag{2.22}$$

Compensation for (2.8) in (2.22), we have

$$\mu_{r+1|r}=\varphi(x)+\frac{l}{ac[1-G(x)]^{n-r}}\int_x^{\gamma}\frac{be^{\varphi(y)}-a}{\left[lG(y)+v\right]}.\left[1-G(y)\right]^{n-r}g(y)dy. \tag{2.23}$$

Simplifying (2.23), we obtain (2.20). Thus (2.1) implies (2.20), i.e. the necessary condition is proved. To prove the sufficient condition, from (2.20) and (1.7), we have

$$\frac{(n-r)}{[1-G(x)]^{n-r}}\int_x^{\gamma}\varphi(y)\left[1-G(y)\right]^{n-r-1}g(y)dy=\varphi(x)+\frac{lb}{ac[1-G(x)]^{n-r}}\int_x^{\gamma}e^{\varphi(y)}$$
$$\times\left[1-G(y)\right]^{n-r-1}N(y)g(y)dy-\frac{l}{c[1-G(x)]^{n-r}}\int_x^{\gamma}N(y)\left[1-G(y)\right]^{n-r-1}g(y)dy. \tag{2.24}$$

Taking the derivative of (2.24) with respect to x, we get (2.8), and integrate it we have (2.1), thus (2.20) implies (2.1). Then, the Theorem is proved.

Special case

Return to (2.17), if we put $l=-1$, $v=1$ we get

$$\mu_{r+1|r}=\varphi(x)-\frac{b}{ac(n-r)}E_{r+1|r}\left[e^{\varphi(y)}\middle|X_r=x\right]+\frac{1}{c(n-r)},\ \alpha<x<y<\beta,$$

it is before doubly truncated case (Table 1).

Theorem 5

Referring to (1.6), (1.7) and (2.2), then

$$\mu_{r|s}=\mu_{r+1|s}-\frac{1}{bcr}\left\{bE_{r|s}\left[\varphi(X_{r:n})V(X_{r:n})\middle|X_{s:n}=y\right]-aE_{r|s}\left[V(X_{r:n})\middle|X_{s:n}=y\right]\right\}, \tag{2.25}$$

where $V(x)$ is defined in (2.5).

Proof

As before in Theorem (1), differentiate (2.2) with respect to x, we have

$$\varphi'(x)=\frac{b\varphi(x)-a}{bc}\frac{lg(x)}{lG(x)+v}. \tag{2.26}$$

Compensation for (2.26) in (2.7), we get

$$\mu_{r|s}=\mu_{r+1|s}-\frac{(s-1)!}{r!(s-r-1)![G(y)]^{s-1}}\int_{\varepsilon}^{y}\left\{\frac{b\varphi(x)-a}{bc}\right\}$$
$$V(x)[G(x)]^{r-1}\left[G(y)-G(x)\right]^{s-r-1}g(x)dx \tag{2.27}$$
$$=\mu_{r+1|s}-\frac{1}{bcr}\int_{\varepsilon}^{y}\left(b\varphi(x)-a\right)V(x)g_{r|s}(x|y)^{r-1}dx$$

Simplifying (2.27), we obtain (2.25). Thus, the Theorem is proved.

Name	[$lG(x)+v$]	$\varphi(x)$	(l,v)	(a,b,c)
Weibull	$e^{-\theta x^p};\alpha\le\varepsilon<x<\gamma\le\beta,\varepsilon=0,\gamma\to\infty$	x^p θx^p	$\left(e^{-\theta\varepsilon^p}-e^{-\theta\gamma^p},e^{-\theta\varepsilon^p}\right)$	(–1,0,θ) (–1,0,1)
Pareto	$\theta^p x^{-p};\ \alpha\le\varepsilon<x<\gamma\le\beta,\ \varepsilon=0,\ \gamma\to\infty$	$\ln(x)$ $\ln[x^{-p}]$	$(\theta^p(\gamma^{-p}-\varepsilon^{-p}),\ \theta^p\varepsilon^{-p})$	$(-\theta,0,p)$ $(-\theta^p,0,-1)$
Power function	$1-\theta^{-p}x^p,\alpha\le\varepsilon<x<\gamma\le\beta,\ \varepsilon=0,\ \gamma\to\infty$	$\ln\left[\frac{x}{\theta}\right]^{-p}$ $\ln[x^p]$	$(\theta^{-p}(\varepsilon^p-\gamma^p),\ 1-\theta^{-p}\varepsilon^p)$	(1,1,1) $(\theta^{-p},1,1)$
Rayleigh	$e^{(-\theta x^2)};a\le\varepsilon<x<\gamma\le\beta$	x^2	$\left(e^{-\theta\gamma^2}+e^{-\theta\varepsilon^2},e^{-\theta\gamma^2}\right)$	$(-1,0,\theta)$
Inverse Weibull	$e^{(-\theta x^{-p})};a\le\varepsilon<x<\gamma\le\beta$	θx^{-p}	$\left(e^{-\theta\gamma^{-p}}+e^{-\theta\varepsilon^{-p}},e^{-\theta\varepsilon^{-p}}\right)$	(–1,0,1)

Table 1: Example of $G(x)=\frac{1}{l}\{[b-ae^{-\varphi(x)}]^c-v\}$ distributions.

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Theorem 6

Referring to (1.6), (1.7), then (2.2) if and only if

$$\mu_{r|r+1}=\varphi(y)-\frac{1}{bcr}\left\{bE_{r|r+1}\left[\varphi(X_{r:n})V(X_{r:n})\middle|X_{r+1:n}=y\right]-aE_{r|r+1}\left[V(X_{r:n})\middle|X_{r+1:n}=y\right]\right\}, \quad (2.28)$$

Where $V(x)$ is defined in (2.5).

Proof

As before in Theorem (2), from (2.26) and (2.11), we have

$$\mu_{r|r+1}=\varphi(y)-\frac{1}{\left[G(y)\right]^{r}}\int_{\varepsilon}^{y}\frac{b\varphi(x)-a}{bc}\frac{lG(x)}{lG(x)+v}\left[G(x)\right]^{r-1}g(x)dx$$
$$=\varphi(y)-\frac{1}{bcr}\int_{\varepsilon}^{y}(b\varphi(x)-a)V(x)g_{r|r+1}(x|y)dx \quad (2.29)$$

Therefore, we get (2.28), then (2.2) implies (2.28). To prove the sufficient condition, from (2.28) and (1.7), we obtain

$$\frac{r}{[G(y)]^{r}}\int_{\varepsilon}^{y}\varphi(x)\left[G(x)\right]^{r-1}g(x)dx=\varphi(y)-$$
$$\frac{1}{bcr}\int_{\varepsilon}^{y}(b\varphi(x)-a)\frac{lG(x)}{lG(x)+v}\left[G(x)\right]^{r-1}g(x)dx \quad (2.30)$$

Taking the derivative, we get (2.26) and we obtain, after integration, (2.2). Thus (2.28) implies (2.2).

Special case

Return to (2.28), then put l=–1, v=1, we get

$$\mu_{r|r+1}=\varphi(y)+\frac{1}{bcr}\{a\,E_{r|r+1}\left[V(X_{r:n})\middle|X_{r+1}=y\right]-b\,E_{r|r+1}\left[\varphi(X_{r:n})V(X_{r:n})\middle|X_{r+1}=y\right],$$
$$\alpha<x<y<ß,$$

it is before doubly truncated case.

Theorem 7

Referring to (1.5), (1.7) and (2.2), then

$$\mu_{s|r}=\mu_{s-1|r}-\frac{l}{bc(n-s+1)}$$
$$\left\{bE_{s|r}\left[\varphi(X_{s:n}x)N(X_{s:n})\middle|X_{r:n}=x\right]-aE_{s|r}\left[N(X_{s:n})\middle|X_{r:n}=x\right]\right\}, \quad (2.31)$$

where $N(y)$ is defined in (2.17).

Proof

As before in Theorem (3), compensation for (2.26) in (2.18), we have

$$\mu_{s|r}=\mu_{s-1|r}-\frac{(n-r)!}{bc(s-r-1)!(n-s+1)![1-G(x)]^{n-r}}\int_{x}^{\gamma}(b\varphi(y)-a)N(y)\left[G(y)-G(x)\right]^{s-r-1}$$
$$\times\left[1-G(y)\right]^{n-s}g(y)dy=\mu_{s-1|r}-\frac{l}{bc(n-s+1)}\{\frac{1}{c}\int_{x}^{\gamma}(b\varphi(y)-a)N(y)g_{s|r}(y|x)dy \quad (2.32)$$

After simplification, we get (2.31). Then (2.2) implies (2.31).

Theorem 8

Referring to (1.5), (1.7), then (2.2) if and only if

$$\mu_{r+1|r}=\varphi(x)+\frac{l}{bc(n-r)}$$
$$\left\{bE_{r+1|r}\left[\varphi(X_{r+1:n})N(X_{r+1:n})\middle|X_{r:n}=x\right]-aE_{r+1|r}\left[N(X_{r+1:n})\middle|X_{r:n}=x\right]\right\}, \quad (2.33)$$

where $N(y)$ is defined in (2.17).

Proof

As before in Theorem (4), compensation for (2.26) in (2.23), we have

$$\mu_{r+1|r}=\varphi(x)+\frac{l}{bc\,[1-G(x)]^{n-r}}$$
$$\int_{x}^{\gamma}\left[b\varphi(y)-a\right]N(y)\left[1-G(y)\right]^{n-r-1}g(y)dy \quad (2.34)$$

Then, we obtain (2.35). Thus (2.2) implies (2.33). Now from (1.7) and (2.33) we get

$$\frac{(n-r)}{[1-G(x)]^{n-r}}\int_{x}^{\gamma}\varphi(y)\left[1-G(y)\right]^{n-r-1}g(y)dy=\varphi(x)+\frac{l}{c[1-G(x)]^{n-r}}\int_{x}^{\gamma}\varphi(y)N(y)$$
$$\times\left[1-G(y)\right]^{n-r-1}g(y)dy-\frac{al}{bc[1-G(x)]^{n-r}}\int_{x}^{\gamma}N(y)\left[1-G(y)\right]^{n-r-1}g(y)dy \quad (2.35)$$

Taking the derivative with respect to x, we obtain (2.26), and integrate it we have (2.2), thus (2.33) implies (2.2).

Hence, the Theorem is proved. Special caseReturn to (2.33), then put l=–1, v=1, we get

$$\mu_{r+1|r}=\varphi(x)-\frac{1}{bc(n-r)}$$
$$\left\{bE_{r+1|r}\left[\varphi(Y)\middle|x\right]-a\right\},\ \alpha<x<y<\beta$$

it is before doubly truncated case (Table 2).

Theorem 9

Referring to (1.6), (1.7) and (2.3), then

$$\mu_{r|s}=\mu_{r+1|s}-\frac{l}{acr}\left\{(b-v)E_{r|s}\left[e^{c\varphi(X_{r:n})}\middle|X_{r:s}=y\right]-a\right\} \quad (2.36)$$

Name	[lG(x) + v]	ϕ(x)	(l,v)	(a,b,c)
Weibull	$\left(e^{-\theta\varepsilon^{p}}-e^{-\theta\gamma^{p}},e^{-\theta\varepsilon^{p}}\right)$	$e^{-x^{p}}$ $e^{-\theta x^{p}}$	$e^{-\theta x^{p}}$	(-1,0,θ)
Power function	$1-\theta^{-p}x^{p}$, $\alpha\le\varepsilon<x<\gamma\le\beta$, $\varepsilon=0$, $\gamma\to\infty$	$\left(\frac{x}{\theta}\right)^{p}$ x^{p}	$(\theta^{-p}(\varepsilon^{p}+\gamma^{p}), 1-\theta^{-p}\varepsilon^{p})$	(1,1,1) $(\theta^{-p}$,1,1)
Rayleigh	$e^{-\theta x^{2}};\alpha\le\varepsilon<x<\gamma\le\beta,$ $\varepsilon=0,\gamma\to\infty$	$e^{-x^{2}}$	$\left(e^{-\theta\gamma^{2}}+e^{-\theta\varepsilon^{2}},e^{-\theta\gamma^{2}}\right)$	(–1,0,θ)
Inverse Weibull	$e^{-\theta x^{-p}};\alpha\le\varepsilon<x<\gamma\le\beta,$ $\varepsilon=0,\gamma\to\infty$	$e^{-\theta x^{-p}}$	$\left(e^{-\theta\gamma^{-p}}+e^{-\theta\varepsilon^{-p}},e^{-\theta\varepsilon^{-p}}\right)$	(0,–1,1)

Table 2: Example of $G(x)=\frac{1}{l}\{[b-a\varphi(x)]^{c}-v\}$ distributions.

Name	$[lG(x)+v]$	$\phi(x)$	(l,v)	(a,b,c)
Power distribution	$1-\theta^{-p}x^p, \alpha\le\varepsilon<x<\gamma\le\beta, \varepsilon=0, \gamma=\theta$	$Ln[1-\theta^{-p}x^p]$	$(\theta^{-p}(\varepsilon^p+\gamma^p), 1-\theta^{-p}\varepsilon^p)$	(1,0,–1)
Weibull	$e^{-\theta x^p}, \alpha\le\varepsilon<x<\gamma\le\beta, \varepsilon=0, \gamma\to\infty$	θx^p	$(e^{-\theta\gamma^p}+e^{-\theta\varepsilon^p}, e^{-\theta\varepsilon^p})$	(–1,0,1)
Burr	$(1+\theta x^p)^{-\gamma}, \alpha\le\varepsilon<x<\gamma\le\beta, \varepsilon=0, \gamma\to\infty$	$ln[1+\theta x^p]$	$(\theta\gamma^{-p})+\theta^{\varepsilon p}+1), 1+\theta^{\varepsilon p})$	(–1,0,γ)
Inverse Weibull	$e^{-\theta x^{-p}}, \alpha\le\varepsilon<x<\gamma\le\beta, \varepsilon=0, \gamma\to\infty$	$e^{-\theta x^{-p}}$	$(e^{-\theta\gamma^p}+e^{-\theta\varepsilon^p}, e^{-\theta\varepsilon^{-p}})$	(–1,0,1)

Table 3: Example of $G(x)=\frac{1}{l}\left\{b-ae^{-c\varphi(x)}-v\right\}$ distributions.

Proof

As given in Theorems (1) and (5), , differentiate (2.3) with respect to x, we have

$$\varphi(x)=\frac{l}{ac}e^{c\varphi(x)}g(x)\cdot \qquad (2.37)$$

Compensation for (2.37) in (2.7), we get

$$\mu_{r|s}=\mu_{r+1|s}-\frac{1}{acr}\int_{\varepsilon}^{y}\left[(b-v)e^{c\varphi(x)}-a\right]g_{r|s}(x|y)dx\ , \qquad (2.38)$$

which gives (2.36). Thus, (2.3) implies (2.36).

Theorem 10

Referring to (1.6) and (1.7), then (2.3) if and only if

$$\mu_{r|r+1}=\varphi(y)-\frac{1}{acr}\left\{(b-v)E_{r|r+1}\left[e^{c\varphi(X_{r:n})}\middle|X_{r+1:n}=y\right]-a\right\}, \qquad (2.39)$$

Proof

As given previously of Theorems (2) and (6), substituting from (2.26) in (2.11), we have

$$\mu_{r|r+1}=\varphi(y)-\frac{1}{acr}\int_{\varepsilon}^{y}e^{c\varphi(x)}\left[b-v-ae^{-c\varphi(x)}\right]g_{r|r+1}(x|y)dx \qquad (2.40)$$

After some simplification, we get (2.36). Then (2.3) implies (2.36). Now from (1.7) and (2.36), we obtain

$$r\int_{\varepsilon}^{y}\varphi(x)\left[G(x)\right]^{r-1}g(x)dx=[G(y)]^r$$

$$\left[\varphi(y)-\frac{(b-v)}{acr}\int_{\varepsilon}^{y}\frac{r\left[G(x)\right]^{r-1}e^{c\varphi(x)}g(x)}{\left[G(y)\right]^r}dx+\frac{1}{cr}\right]$$

Taking the derivative with respect to y, we get

$$\frac{l\,g(y)}{l\,\mathrm{G}(y)}=\frac{ace^{-c\varphi(y)}\varphi(y)}{b-v-ae^{-c\varphi(y)}}\ . \qquad (2.41)$$

Integrate (2.41), we obtain (2.3).

Special case

Return to (2.36), then put l=−1, v=1, we get

$$\mu_{r|r+1}=\varphi(y)-\frac{1}{acr}\left\{(b-1)E_{r|r+1}\left[e^{c\varphi(X_{r:n})}\middle|X_{r+1:n}=y\right]-a\right\}, \alpha<x<y<\beta$$

it is before doubly truncated case.

Theorem 11

Referring to (1.5), (1.7) and (2.3), then

$$\mu_{s|r}=\mu_{s-1|r}-\frac{l}{ac(n-s+1)}\{(l-b+v)E_{s|r}\left[e^{c\varphi(x_{s:n})}\middle|X_{r:n}=x\right]+a\ . \qquad (2.42)$$

Proof

As previously in Theorems (3) and (7), from (2.37) in (2.18), we have

$$\mu_{s|r}=\mu_{s-1|r}-\frac{1}{ac(n-s+1)}\int_{x}^{\gamma}\left[(l-b+v)e^{c\varphi(y)}+a\right]g_{s|r}(y|x)dy\ ,$$

which gives (2.42).

Theorem 12

Referring to (1.5) and (1.7), then (2.3) if and only if

$$\mu_{r+1|r}=\varphi(x)+\frac{(l-b+v)}{ac(n-r)}E_{r+1|r}\left[e^{c\varphi(X_{r+1:n})}\middle|X_{r:n}=x\right]+\frac{1}{c(n-r)}\ .(2.43)$$

Proof

Similarly as given in Theorems (4) and (8), we easily prove it.

Special case

Return to (2.43), then put l=−1, v=1, we get

$$\mu_{r+1|r}=\varphi(x)-\frac{b}{ac(n-r)}E_{r+1|r}\left[e^{c\varphi(X_{r+1:n})}\middle|X_{r:n}=x\right]+\frac{1}{c(n-r)},\ \alpha<x<y<\beta\ .$$

The relation before doubly truncated case (Table 3).

Conclusion

It was obtained recurrence relations based on order statistics without truncated and doubly truncated, and have been getting function of various distributions new by using certain parameters.

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References

1. David HA (1981) Order Statistics. (2ndedn) John Wiley, New York.
2. Balakrishnan N, Cohen AC (1991) Order Statistics and Inference: Estimation Methods. Academic Press, San Diego.
3. Arnold BC, Balakrishnan N, Nagaraja HN (1992) A first Course in order Statistics. John Wiley, New York.
4. David HA (1995) On recurrence relations for order statistics. Statistics and Probability Letters 24: 133-138.
5. David HA, Nagaraja HN (2003) Order Statistics. (3rdedn) John Wiley, New York.
6. Mahmoud MAW, Sultan KS, Amer SM (2003) Order statistics from inverse Weibull distribution and associated inference. Comput Statist Data Analysis 42: 49-163.
7. Mahmoud MAW, Sultan KS, Amer SM (2003) Order statistics from inverse Weibull distribution and characterizations. Metron LXI 3: 389-401.
8. Balakrishnan N, Sultan KS (1998) Recurrence relations and identities for moments of order statistics. In: Balakrishnan N and Rao CR (eds.) Handbook of statistics16: 149-228.
9. El-Din M, Mahmoud MM, Abu-Youssef MAW, Sultan SE (1997) Order statistics from the doubly truncated linear-exponential distribution and its characterizations. Commun Statist- Simul Comput 26: 281-290.
10. Abu-Youssef SE (2003) On characterization of certain distribution of record values. Applied Mathematics and Computation 145: 445-450.

11. Abd-El-Mougod GA (2005) On Record Values and Its Applications. Sohag University, Egypt.

12. Shawky AI, Abu-Zinadah HH (2008) General recurrence relations and characterizations of certain distributions based on order statistics. Journal of Statistical Theory and Applications 7: 93-117.

13. Shawky AI, Bakoban RA (2009) Conditional expectation of certain distributions of record values. Int Journal of Math Analysis 3: 829-838.

14. Pushkarna N, Saran J, Tiwari R (2012) Recurrence relations for higher moments of order statistics from doubly truncated exponential distribution. International Mathematical Forum, 7(4), 193-201.

15. Joshi PC (1979) A note on the moments of order statistics from doubly truncated exponential distribution. Annals of the Institute of Statistical Mathematics 31: 321-324.

16. Balakrishnan N, Joshi PC (1984) Product moments of order statistics from the doubly truncated exponential distribution. Naval Research Logistics Quarterly: 31: 27-31.

17. Khan AH, Ali MM (1987) Characterization of probability distributions through higher order gap. Communications in Statistics-Theory and Methods 16: 1281-1287.

18. Ahmad AA (2001) Moments of order statistics from doubly truncated continuous distribution. Statistics 35: 479-494.