

The Finite Difference Methods for Parabolic Partial Differential Equations

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Abstract

In this paper finite difference methods will be used to solve both one and two dimensional heat equations which are the well-known partial differential equations.

Keywords: Parabolic; Heat equation; Finite difference; Bender-Schmidt; Crank-Nicolson

Introduction

Parabolic partial differential equations

The well-known parabolic partial differential equation is the one dimensional heat conduction equation [1]. The solution of this equation is a function $u(x,t)$ which is defined for values of x from 0 to l and for values of t from 0 to ∞ [2-4]. The solution is not defined in a closed domain but advances in an open ended region from initial values, satisfying the prescribed boundary conditions (Figure 1). In general the study of pressure waves in a fluid, propagation of heat and unsteady state problems lead to parabolic type of equations [5].

Solution of one-dimensional Heat conduction equation

Consider the heat conduction which is of the form [6]

$$cu_t = u_{xx} \quad (1)$$

where C is a constant. Let the plane (x,t) be divided into smaller rectangles by means of the sets of lines [7]

$$x = ih, i = 0, 1, 2, \dots$$

$$t = jk, j = 0, 1, 2, \dots$$

Using the finite difference approximations [8-11]

$$u_t = \frac{u_{i,j+1} - u_{i,j}}{k} \quad (2)$$

and

$$u_{xx} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \quad (3)$$

Hence the finite difference analogue in eqn. (1) is

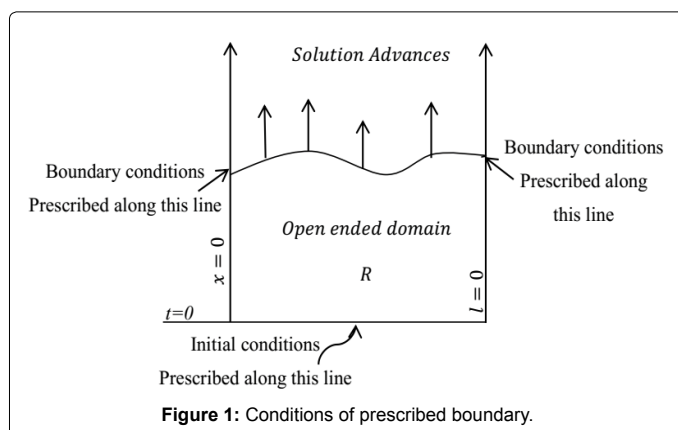


Figure 1: Conditions of prescribed boundary.

$$\frac{c}{k}(u_{i,j+1} - u_{i,j}) = \frac{1}{h^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) \quad (4)$$

Which can be rewritten as

$$u_{i,j+1} = u_{i,j} + \lambda(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$$

where $\lambda = \frac{ck}{h^2}$ this formula expresses the unknown function value at the $(i,j+1)^{th}$ interior point in terms of the known function values and hence it is called the explicit formula. It can be shown that this formula is valid only [10] for $0 < \lambda < \frac{1}{2}$. For $\lambda = \frac{1}{2}$ becomes in eqn. (4)

$$u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j}) \quad (5)$$

The above formula in eqn. (5) is called Bender-Schmidt recurrence relation. In formula in eqn. (4), we have used the function values along the j^{th} row only in the approximation of u_{xx} is replaced by the [11] average of its finite difference approximations on the j^{th} and $(j+1)^{th}$ rows. Thus

$$u_{xx} = \frac{1}{2} \left(\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2} \right)$$

and hence in eqn. (1) is replaced by

$$\frac{c}{k}(u_{i,j+1} - u_{i,j}) = \frac{1}{2h^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1})$$

This implies

$$-\lambda u_{i-1,j+1} + (2 + 2\lambda)u_{i,j+1} - \lambda u_{i+1,j+1} = \lambda u_{i-1,j} + (2 - 2\lambda)u_{i,j} + \lambda u_{i+1,j} \quad (6)$$

where $\lambda = \frac{ck}{h^2}$, on the left side in eqn. (6) we have three unknowns and on the right side all the three quantities are known. The above formula in eqn. (6) which is an implicit scheme is called *Crank-Nicolson* formula and it is convergent for all finite values of λ . If there are N internal mesh points on each row, then the formula in eqn. (6) gives N simultaneous equations for the N unknowns in terms of the given boundary values.

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Similarly the internal mesh points on all rows can be calculated [12].

Example 1: Use the Bender-Schmidt recurrence relation to solve the equation $u_{xx} = 2u_t$. With the conditions $u(x,0) = 4x - x^2, u(0,t) = u(4,t) = 0$ Solution Taking $h=1$ we obtain [13]

$$k = \frac{1}{2} h^2 c = 1$$

Also $u(0,0)=0, u(1,0)=3, u(2,0)=4, u(3,0)=3$ and $u(4,0)=0$. For this the first time step, $k=1$.

For this the first time step, $k=1$.

Using Bender-Schmidt recurrence relation, we obtain

$$u_{1,1} = \frac{1}{2}(0 + 4) = 2, u_{2,1} = \frac{1}{2}(3 + 3) = 3, u_{3,1} = \frac{1}{2}(4 + 0) = 2$$

For $k=2$, we have

$$u_{1,2} = \frac{1}{2}(0 + 3) = 1.5, u_{2,2} = \frac{1}{2}(2 + 2) = 2, u_{3,2} = \frac{1}{2}(3 + 0) = 1.5$$

For $k=3$ we have

$$u_{1,3} = \frac{1}{2}(0 + 2) = 1, u_{2,3} = \frac{1}{2}(1.5 + 1.5) = 1.5, u_{3,3} = \frac{1}{2}(2 + 0) = 1$$

For $k=4$ we have

$$u_{1,4} = \frac{1}{2}(0 + 1.5) = 0.75, u_{2,4} = \frac{1}{2}(1 + 1) = 1, u_{3,4} = \frac{1}{2}(1.5 + 0) = 0.75$$

Similarly with $k=5$ we obtain

$$u_{1,5} = \frac{1}{2}(0 + 1.0) = 0.5, u_{2,5} = \frac{1}{2}(0.75 + 0.75) = 0.75, u_{3,5} = \frac{1}{2}(1 + 0) = 0.5$$

The computation can be continued to any number of time steps.

Example 2: Solve the equation $u_t = u_{xx}$, subject to the conditions, $u(x,0) = \sin \pi x, 0 \leq x \leq 1, u(0,t) = u(1,t) = 0$.

Using

(a) Bender-Schmidt Method

(b) Crank-Nicolson Method

Carry out the computations for two levels, taking $h = \frac{1}{3}, k = \frac{1}{36}$

Solution

Here $c=1, h = \frac{1}{3}$ and $k = \frac{1}{36}$ so that $\lambda = \frac{k}{(ch^2)} = \frac{1}{4}$

Also $u_{1,0} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}, u_{2,0} = \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$ and all boundary values

are zero in Figure 2.

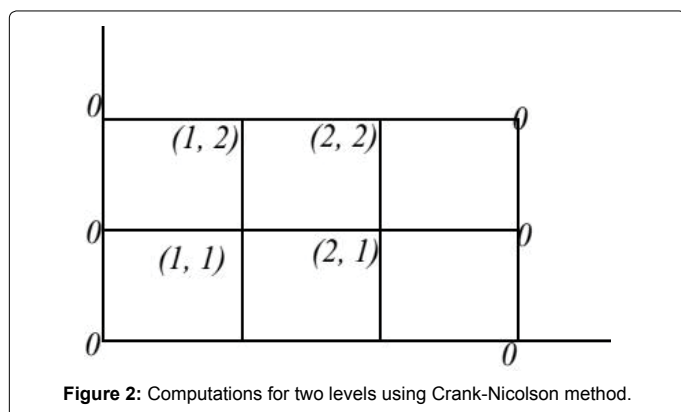


Figure 2: Computations for two levels using Crank-Nicolson method.

(a) Using Bender-Schmidt formula

Using in eqn. (4) of this section

we have $u_{i,j+1} = \lambda u_{i-1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i+1,j}$

This becomes

$$u_{i,j+1} = \frac{1}{4}(u_{i-1,j} + 2u_{i,j} + u_{i+1,j})$$

For $i=1,2; j=0$

$$u_{1,1} = \frac{1}{4}(u_{0,0} + 2u_{1,0} + u_{2,0}) = \frac{1}{4}\left(0 + \frac{\sqrt{3}}{2} + \frac{3}{2}\right) = 0.65$$

$$u_{2,1} = \frac{1}{4}(u_{1,0} + 2u_{2,0} + u_{3,0}) = \frac{1}{4}\left(\frac{\sqrt{3}}{2} + 2 + \frac{\sqrt{3}}{2} + 0\right) = 0.65$$

For $i=1,2; j=1$

$$u_{1,2} = \frac{1}{4}(u_{0,1} + 2u_{1,1} + u_{2,1}) = 0.49$$

$$u_{2,2} = \frac{1}{4}(u_{1,1} + 2u_{2,1} + u_{3,1}) = 0.49$$

(b) By using Crank-Nicolson method

From eqn. (6) above we have

$$-\frac{1}{4}u_{i-1,j+1} + \frac{5}{2}u_{i,j+1} - \frac{1}{4}u_{i+1,j+1} = -\frac{1}{4}u_{i-1,j} + \frac{3}{2}u_{i,j} + \frac{1}{4}u_{i+1,j}$$

For $i=1,2; j=0$

$$-u_{0,1} + 10u_{1,1} - u_{2,1} = u_{0,0} + 6u_{1,0} + u_{2,0}$$

$$\Rightarrow 10u_{1,1} - u_{2,1} = 7\frac{\sqrt{3}}{2}$$

$$-u_{1,1} + 10u_{2,1} - u_{3,1} = u_{1,0} + 6u_{2,0} + u_{3,0}$$

$$\Rightarrow -u_{1,1} + 10u_{2,1} = 7\frac{\sqrt{3}}{2}$$

Solving these equations, we find

$$u_{1,1} = u_{2,1} = 0.67$$

For $i=1,2; j=1$

$$-u_{0,2} + 10u_{1,2} - u_{2,2} = u_{0,1} + 6u_{1,1} + u_{2,1}$$

$$\Rightarrow 10u_{1,2} - u_{2,2} = 4.69$$

$$-u_{1,2} + 10u_{2,2} - u_{3,2} = u_{1,1} + 6u_{2,1} + u_{3,1}$$

$$\Rightarrow -u_{1,2} + 10u_{2,2} = 4.69$$

Again solving the above two equation we can obtain

$$u_{1,2} = u_{2,2} = 0.52$$

Solution of two dimensional heat equations

Two dimensional heat equations is an equation of the form

$$u_t = c^2(u_{xx} + u_{yy}).$$

The method employed for the solution of one dimensional heat equation can be readily extended to the solution two dimensional heat equations in eqn. (7).

Consider a square region $0 \leq x \leq y \leq a$ and assume that u is known at all points within and on the boundary of this square.

If h be the step size then a mesh point

$(x,y,t) = (ih,jh,nl)$ may be denoted as (i,j,n) .

Replacing the derivatives in eqn. (7) by their finite difference approximations, we get

$$\frac{u_{i,j,n+1} - u_{i,j,n}}{l} = \frac{c^2}{h^2} \left[(u_{i-1,j,n} - 2u_{i,j,n}) + (u_{i,j-1,n} + u_{i,j+1,n}) \right] \quad (8)$$

$$u_{i,j,n+1} = u_{i,j,n} + \lambda (u_{i-1,j,n} + u_{i+1,j,n} + u_{i,j-1,n} + u_{i,j+1,n} - 4u_{i,j,n})$$

Where $\lambda = \frac{lc^2}{h^2}$. This equation needs the five points available on the n^{th} plane (Figure 3).

The computation process consists point by point evaluation in the $(n+1)^{\text{th}}$ plane using the points on the n^{th} plane. This method is known as ADE (Alternating Direct Explicit) method.

Example: Solve the equation $u_t = u_{xx} + u_{yy}$.

Subject to the initial conditions $u(x,y,0) = \sin 2\pi x \sin 2\pi y, 0 \leq x,y \leq 1$ and the conditions $u(x,y,t) = 0, t \geq 0$, on the boundaries using ADE method $h = \frac{1}{3}$ and $\lambda = \frac{1}{3}$ (Calculate the results for one time level).

Solution: In eqn. (8) becomes

$$u_{i,j,n+1} = u_{i,j,n} + \frac{1}{8} (u_{i-1,j,n} + u_{i+1,j,n} + u_{i,j-1,n} + u_{i,j+1,n} - 4u_{i,j,n})$$

$$u_{i,j,n+1} = \frac{1}{2} u_{i,j,n} + \frac{1}{8} (u_{i-1,j,n} + u_{i+1,j,n} + u_{i,j-1,n} + u_{i,j+1,n}) \quad (*)$$

The mesh points and the computation model is given in Figure 4 below

At the zeroth level ($n=0$), the initial and boundary conditions are

$$u_{i,j,0} = \sin \frac{2\pi i}{3} \sin \frac{2\pi j}{3}$$

And

$$u_{i,0,0} = u_{0,j,0} = u_{3,j,0} = u_{i,3,0} = 0; i,j=0,1,2,3$$

Now we can calculate the mesh values at the first level.

For $n=0$, from (*) we obtain

$$u_{i,j,1} = \frac{1}{2} u_{i,j,0} + \frac{1}{8} (u_{i-1,j,0} + u_{i+1,j,0} + u_{i,j-1,0} + u_{i,j+1,0}) \quad (**)$$

(i) Putting $i=j=1$ in (**) above

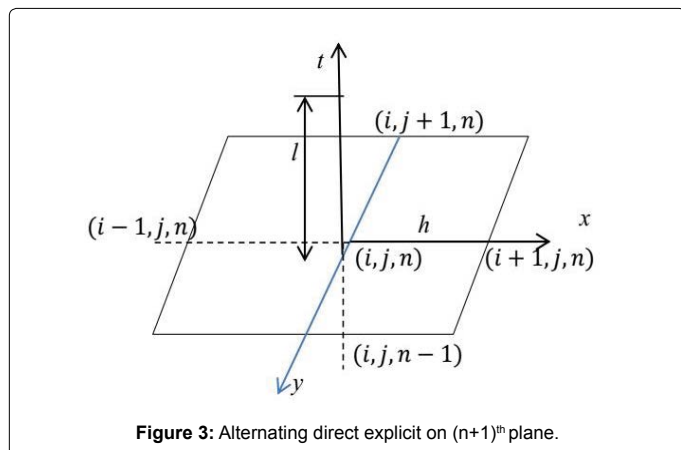


Figure 3: Alternating direct explicit on $(n+1)^{\text{th}}$ plane.

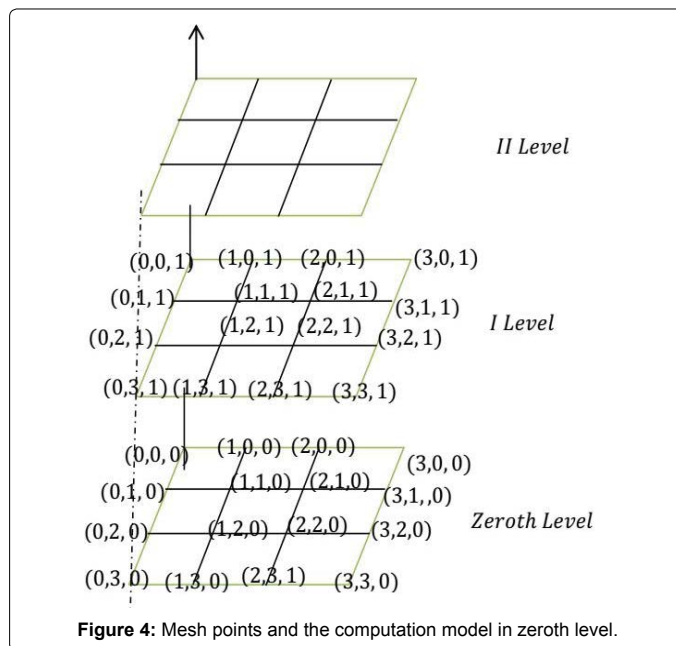


Figure 4: Mesh points and the computation model in zeroth level.

$$\begin{aligned} u_{1,1,1} &= \frac{1}{2} u_{1,1,0} + \frac{1}{8} (u_{0,1,0} + u_{2,1,0} + u_{1,0,0} + u_{1,2,0}) \\ &= \frac{1}{2} \left(\sin \frac{2\pi}{3} \right)^2 + \frac{1}{8} \left(0 + \sin \frac{4\pi}{3} \sin \frac{2\pi}{3} + \sin \frac{2\pi}{3} \sin \frac{4\pi}{3} + 0 \right) \\ &= \frac{3}{8} + \frac{1}{8} \left(-\frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2} \right) = \frac{3}{16} \end{aligned}$$

(ii) Putting $i=2,j=1$ in (**) above

$$\begin{aligned} u_{2,1,1} &= \frac{1}{2} u_{2,1,0} + \frac{1}{8} (u_{0,1,0} + u_{3,1,0} + u_{2,2,0} + u_{2,0,0}) \\ &= \frac{1}{2} \sin \frac{4\pi}{3} \sin \frac{2\pi}{3} + \frac{1}{8} \left[\left(\sin \frac{2\pi}{3} \right)^2 + 0 + \left(\sin \frac{4\pi}{3} \right)^2 + 0 \right] \\ &= -\frac{1}{2} \left(\frac{\sqrt{3}}{2} \right)^2 + \frac{1}{8} \left[\left(\frac{\sqrt{3}}{2} \right)^2 + \left(-\frac{\sqrt{3}}{2} \right)^2 \right] = -\frac{3}{16} \end{aligned}$$

(iii) Putting $i=1,j=2$ in (**) above

$$\begin{aligned} u_{1,2,1} &= \frac{1}{2} u_{1,2,0} + \frac{1}{8} (u_{0,2,0} + u_{3,2,0} + u_{1,3,0} + u_{1,1,0}) \\ &= \frac{1}{2} \sin \frac{2\pi}{3} \sin \frac{4\pi}{3} + \frac{1}{8} \left[0 + \left(\sin \frac{4\pi}{3} \right)^2 + 0 + \left(\sin \frac{2\pi}{3} \right)^2 \right] \\ &= \frac{3}{8} + \frac{1}{8} \left(\frac{3}{4} + \frac{3}{4} \right) = -\frac{3}{16} \end{aligned}$$

(iv) Putting $i=j=2$ in (**) above

$$\begin{aligned} u_{2,2,1} &= \frac{1}{2} u_{2,2,0} + \frac{1}{8} (u_{1,2,0} + u_{3,2,0} + u_{2,3,0} + u_{2,1,0}) \\ &= \frac{1}{2} \left(\sin \frac{4\pi}{3} \right)^2 + \frac{1}{8} \left(\sin \frac{2\pi}{3} \sin \frac{4\pi}{3} + 0 + 0 + \sin \frac{4\pi}{3} \sin \frac{2\pi}{3} \right) \\ &= \frac{3}{8} + \frac{1}{8} \left(-\frac{3}{4} - \frac{3}{4} \right) = -\frac{3}{16} \end{aligned}$$

Similarly the mesh values at the second and higher levels can be calculated.

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