**Appendix**

The appendix is organized as follows. In Section A.1, we consider a general stochastic approximation Monte Carlo algorithm, and give a theorem for its convergence which has been proved in the literature.

In Section A.2, we proved Theorem 2.1.

**Convergence of a General Stochastic Approximation Monte Carlo Algorithm**

Let  be the collection of the samples generated by a MH kernel at iteration ***t,***  be the invariant distribution of the MH kernel, , and . The SAMC algorithm can then be expressed in a more general form by replacing (7) by (28),

 (28)

The convergence of the general stochastic approximation Monte Carlo algorithm is analyzed by [14] under the following conditions.

**Conditions on the step-sizes**

*(A1)* The sequence  is non-increasing, positive and satisfies the condition (3).

**Drift conditions on the transition kernel :**For a function *g : X* → *Rd*, define the norm



and define the set.

Let *Pθ* be the joint transition kernel for generating the samples ***x*** at each iteration by ignoring the subscript ***t***. Let χ be the sample space, let A be a measurable set belong to *Bx*, and *Bx* is the σ -algebra generated by all subsets of χ.

(A2) The transition kernel *Pθ* is irreducible and aperiodic for any θ∈ Θ, . There exist a function V:X→[1,∞) and constants α ≥ 2 and β ∈(0,1) such that,

(i) For any θ ∈ Θ, there exist a set C ⊂ X, an integer **/**, constants   
 0*< λ <* 1*,b,ς,δ >0* and a probability measure *v* such that,

 (29)

 (30)

 (31)

(ii) There exists a constant *c*1 such that for all x ∈ *X* and *θ*,*θ’* ∈ Θ,

* || *H* (*θ,* *X*) || ≤ *c*1*V* (*x*). (32)
* || *H* (*θ,* *X*) - *H* (*θ’,* *X*) || ≤ *c*1*V* (*x*) || *θ - θ’ ||β* . (33)

(iii) There exists a constant *c*2 such that for all *θ*,*θ’* ∈ Θ,

* *||* *Pθ g - Pθ ,g ||v* ≤*c2 || g ||v |θ - θ’ |β,∀ g ∈ Lv*  (34)

 (35)

**Lyapunov condition:** on *h*(*θ*) Let *L*={θ∈Θ : *h*(θ)-0} *h*(θ) Let 

(A3) The function  is continuous, and there exists a continuously differentiable function such that  and

 for any compact set.

**Convergence of the General SAMC Algorithm**

Let  denote the probability measure of the Markov chain {(*xt,θ*t)}, started in(*x0,θ*0)}, and implicitly defined by the sequences {γt}. Also define .

**Theorem A.1** (Liang, 2009) [16] Assume the conditions (A1),(A2) and (A3) hold, and  Let the sequence {θt} be defined as in (28). Then for all



**A.2 Proof of Theorem 2.1**

It follows from Theorem (A.1), Theorem (2.1) can be proved by verifying that Pop-SAMC satisfies the conditions (A1). to(A3). .

(A1). It is obvious that this condition is satisfied by the sequence as specified in (4).

(A2). Let, which can be regarded as a sample produced by  independent Markov chains on the product space  with the transition kernel



where  denotes a one-step MH kernel at a given value of. Under the assumptions that both Θ and χ are compact and the proposal distribution is local positive, it has been shown in Liang et al. (2007) that Pθ(x,y), satisfies the drift condition (A2). In what follows, we will show that Pθ(x,y), also satisfies (A2). To simplify notations, in what follows we will drop the subscript *t,* denoting ***xt*** by ***x*** and  by 

Roberts and Tweedie [11] (Theorem 2.2) showed that if the target distribution is bounded away from 0 and ∞ on every compact set of its support χ, then the MH chain with a proposal distribution satisfying the local positive condition is irreducible and aperiodic, and every nonempty compact set is small. It follows from this result that Pθ(x,y), is irreducible and aperiodic, and thus Pθ(x,y), is also irreducible and aperiodic.

Since χ is compact, Roberts and Tweedie's result implies that χ is a small set and the minorisation condition holds on χ for the kernel Pθ(x,y), ; i.e., there exists an integer,

a constant δ, and a probability measure  such that



Therefore,



where  and .This verifies condition (31) by setting  Thus, for any θ ∈ Θ the following conditions hold



(36)



by choosing , and α ≥ 2. These conclude that (A2-i) is satisfied.

For Pop-SAMC, we have, where  is an indicator vector of the subregion that x(i) belongs to. Since each component of  takes a value between 0 and 1, there exists a constant  such that for any θ ∈ Θ and all x ∈ χ,

 (37)

Also, H(θ,x) does not depend on $\theta$ for a given sample ***x***. Hence,  for all, and the following condition holds,

 (38)

for all Equations (37) and (38) imply that (A2-A2-ii) is satisfied by choosing  and v(x)=1

In [2], it has shown for the single chain MH kernel that there exists a constant *C2* such that

 (39)

for any measurable set A ⊂ χ. Therefore, there exists a constant *C3* such that















which implies that (34) is satisfied. Condition (A2-iii) is then satisfied by choosing *v(x)=1* and ***β =1.***

(A3) This condition can be verified as in [16]. The proof is re-produced below.

Since the invariant distribution of the kernel *Pθ (x,y)*  is *fθ (x),* we have for any fixed θ,

 (40)

Where H(i)(θ,x), denotes the th component of H(θ,x),  and  Thus, we have



It follows from (40) that h***(θ)*** is a continuous function of ***θ.*** Let  As shown below, ***v(θ)*** has continuous partial derivatives of the first order.

Solving the system of equations formed by (40), we have



where  can be determined by imposing a constraint on S. For example, setting S=1. leads to that C = 0 It is obvious that  is nonempty and ***v(θ)=0*** for every 

To verify the conditions related to  we have the following calculations.



(41)



for *i, j=1,…,m* and *i ≠ j.*



 (42)



For ***i= 1,…,m,*** where  Thus, we have



 (43)



where  denotes the variance of the discrete distribution defined in the following table,

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
| Prob. |  |  |  |

If otherwise, Therefore, for any compact set The proof is completed.