A . Appendix

We will show that following some algebraic manipulations and use of identities we can write the expressions found for the RLFI and RLFD also in terms of the hypergeometric functions. For starters, we will describe the steps for eqn.(28) and the others are done similarly,

So we have that

$$\left[{}_{d}{+}J_t^{\alpha}\left(x-d\right)^{\beta}\right](t)=\sum_{k=0}^{\infty}\frac{\Gamma(\beta+1)\in^{\beta-k}\left(t-d\right)^{\alpha+k}}{\Gamma(\beta-k+1)\Gamma(\alpha+k+1)}$$

$$=\Gamma(\beta+1)\in^{\alpha+\beta}\left(\frac{t-d}{\in}-1\right)^{\alpha}\sum_{k=0}^{\infty}\frac{\left(\frac{t-d}{\in}-1\right)^{k}}{\Gamma(\beta-k+1)\Gamma(\alpha+k+1)}.$$

To simplify the notation we introduce $z=\frac{t-d}{\in}>0$ *and* $\left[{}_{d}{+}J_t^{\alpha}\left(x-d\right)^{\beta}\right](t)=J_{d+}^{\alpha}(t)$ .Hence

$$J_{d+}^{\alpha}(t)=\Gamma(\beta+1)\in^{\alpha+\beta}(z-1)^{\alpha}\sum_{k=0}^{\infty}\frac{(z-1)^{k}}{\Gamma(\beta-k+1)\Gamma(\alpha+k+1)}. \qquad (72)$$

Then using the Pochhammer symbols notation and the identity

$$\frac{(-\beta)_k}{(\alpha+1)_k}=(-1)^k\frac{\Gamma(\beta+1)}{\Gamma(\beta-k+1)}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+k+1)},$$

which implies

$$\frac{1}{\Gamma(\beta-k+1)\Gamma(\alpha+k+1)}=\frac{(-1)^k}{\Gamma(\beta+1)\Gamma(\alpha+1)}\frac{(-\beta)_k}{(\alpha+1)_k}, \qquad (73)$$

we have, after substituting in eqn.(73) in eqn.(72) that

$$J_{d+}^{\alpha}(t)=\frac{\in^{\alpha+\beta}(z-1)^{\alpha}}{\Gamma(\alpha+1)}\sum_{k=0}^{\infty}\frac{(-1)^k(-\beta)_k}{(\alpha+1)_k}(z-1)^k,$$

and since $(1)_k=k!$ we have

$$J_{d+}^{\alpha}(t)=\frac{\in^{\alpha+\beta}(z-1)^{\alpha}}{\Gamma(\alpha+1)}\sum_{k=0}^{\infty}\frac{(-1)_k(-\beta)_k}{(\alpha+1)_k}\frac{(1-z)^k}{k!}. \qquad (74)$$

Now, we recall that the hypergeometric function is defined by the series

$${}_2F_1(a,b;c;\xi)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{\xi^k}{k!},$$

thus in eqn.(74) can be written in the form

$$J_{d+}^{\alpha}(t)=\frac{\in^{\alpha+\beta}(z-1)^{\alpha}}{\Gamma(\alpha+1)}\,{}_2F_1(1,-\beta;\alpha+1;1-z) \qquad (75)$$

We can express this hypergeometric function (conveniently) as a series if, $\alpha+\beta\neq\pm m, m\in$Nand $|\arg(z)|<\pi$only if, + and we can rewrite this hypergeometric function as [37]

$$\begin{aligned}{}_2F_1(1,-\beta;\alpha+1;1-z)&=\frac{\Gamma(\alpha+1)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta+1)}\,{}_2F_1(1,-\beta;1-\alpha-\beta;z)+\\&\frac{\Gamma(\alpha+1)\Gamma(-\alpha-\beta)z^{\alpha+\beta}}{\Gamma(-\beta)}\,{}_2F_1(\alpha,\alpha+\beta+1;\alpha+\beta+1;\mathrm{z}).\end{aligned} \qquad (76)$$

Substituting in eqn.(76) in eqn.(75) we have

$$J_{d+}^{\alpha}(t)=\frac{\in^{\alpha+\beta}(z-1)^{\alpha}\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta+1)}\,{}_2F_1(1,-\beta;1-\alpha-\beta;z)$$
$$+\frac{\in^{\alpha+\beta}(z-1)^{\alpha}\Gamma(-\alpha-\beta)z^{\alpha+\beta}}{\Gamma(-\beta)}\,{}_2F_1(\alpha,\alpha+\beta+1;\alpha+\beta+1;z) \qquad (77)$$

We can continue simplifying this last expression, using some identities for the hypergeometric functions. First, for the second hypergeometric function in eqn.(77), we have:

$$\begin{aligned}{}_2F_1(\alpha,\alpha+\beta+1;z)&=\sum_{k=0}^{\infty}(\alpha)k\frac{z^k}{k!}\\&=\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)k!}z^k\\&=\sum_{k=0}^{\infty}(-1)^k\binom{\alpha+k-1}{k}(-z)^k\\&=(1-z)-\alpha,|z|<1.\end{aligned} \qquad (78)$$

While for the first hypergeometric function in eqn.(77) we use the relation known as Euler transformation

$${}_2F_1(a;\underline{b};c;z) = (1\text{-}z)^{c\text{-}a\text{-}b}\,{}_2F_1(c\text{-}a;c\text{-}b,c;z),$$

and identify that

$$c-a=1-\alpha-\beta-1=-\alpha-\beta,$$
$$c-b=1-\alpha-\beta+\beta=1-\alpha,$$
$$c-a-b=1-\alpha-\beta-1+\beta=-\alpha,$$

Therefore,

$${}_2F_1(1,-\beta;1-\alpha-\beta;z)=(1-z)^{-\alpha}\ {}_2F_1(-\alpha-\beta;1-\alpha,1-\alpha-\beta;z)$$

and it follows that in eqn.(77) assume the following form

$$\mathrm{J}_{d}^{\alpha}+(t)=\frac{(-1)^{\alpha}\Gamma(-\alpha-\beta)}{\Gamma(-\beta)}(t-d)^{\alpha+\beta}$$
$$+\frac{(-1)^{\alpha}\Gamma(\alpha+\beta)\varepsilon^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\alpha+\beta+1)}\,{}_2F_1\left(-\alpha-\beta,1-\alpha;1-\alpha-\beta;\frac{t-d}{\varepsilon}\right). \qquad (79)$$

In a similar fashion in eqn.(27), calling $\left[{}_aJ_t^{\alpha}(x-d)^{\beta}\right](t)=J_a^{\alpha}(t)$ that

$$\mathrm{J}_{d}^{\alpha}+(t)=\frac{(-1)^{\alpha}\Gamma(-\alpha-\beta)}{\Gamma(-\beta)}(t-d)^{\alpha+\beta}$$
$$+\frac{(-1)^{\alpha}\Gamma(\alpha+\beta)(a-d)^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\alpha+\beta+1)}\,{}_2F_1\left(-\alpha-\beta,1-\alpha;1-\alpha-\beta;\frac{t-d}{a-d}\right). \qquad (80)$$

In a similar fashion for Eq.(67), calling $\left[{}_{d+}D_t^{\alpha}(x-d)^{\beta}\right](t)=D_{d+}^{\alpha}(t)$ we have

$$D_{d}^{\alpha}+\left(t\right)=\frac{\left(-1\right)^{\alpha}\Gamma\left(\alpha-\beta\right)}{\Gamma\left(-\beta\right)}\left(t-d\right)^{-\alpha+\beta}$$
$$+\frac{\left(-1\right)^{-\alpha}\Gamma\left(-\alpha+\beta\right)\varepsilon^{-\alpha+\beta}}{\Gamma\left(-\alpha\right)\Gamma\left(-\alpha+\beta+1\right)}\,{}_{2}F_{1}\left(\alpha-\beta,1+\alpha;1+\alpha-\beta;\frac{t-d}{\varepsilon}\right). \tag{81}$$

In a similar fashion in eqn.(66), calling $\left[{}_{d+}D_{t}^{\alpha}\left(x-d\right)^{\beta}\right]\left(t\right)=D_{d+}^{\alpha}\left(t\right)$ we have

$$D_{d}^{\alpha}\left(t\right)=\frac{\left(-1\right)^{\alpha}\Gamma\left(\alpha-\beta\right)}{\Gamma\left(-\beta\right)}\left(t-d\right)^{-\alpha+\beta}$$
$$+\frac{\left(-1\right)^{-\alpha}\Gamma\left(-\alpha+\beta\right)\varepsilon^{-\alpha+\beta}}{\Gamma\left(-\alpha\right)\Gamma\left(-\alpha+\beta+1\right)}\,{}_{2}F_{1}\left(\alpha-\beta,1+\alpha;1+\alpha-\beta;\frac{t-d}{a-d}\right). \tag{82}$$